Units of K-theory and their applications in operator algebras.

Ulrich Pennig[1]

11th September 2026

Interactions between operator algebras, K-theory and homotopy theory

[1]Cardiff University
PennigU@cardiff.ac.uk

Topological K-theory and its units

Properties of complex topological K-theory

- $K^0(X)$ is the group completion of $(\mathcal{V}ect_{\mathbb{C}}(X), \oplus)$, (for X a finite CW-complex)
- $K^1(X) := [X, U]$, where $U = \operatorname{colim}_n U(n)$,
- $X \mapsto K^*(X)$ is a cohomology theory,
- it has a multiplicative structure induced by $\otimes$,
- Bott periodicity: $K^n(X) \cong K^{n+2}(X)$.

Theorem (Joachim, Higson-Guentner)

Topological K-theory is represented by an E_∞-ring spectrum KU (even an orthogonal commutative ring spectrum).

Topological K-theory and its units (continued)

The space of units $GL_1(KU)$ is defined by the pullback diagram

$$\begin{tikzcd}
GL_1(KU) \arrow[r] \arrow[d] & \Omega^\infty KU \simeq BU \times \mathbb{Z} \arrow[d] \\
\mathbb{Z}/2\mathbb{Z} \cong GL_1(\pi_0(\Omega^\infty KU)) \arrow[r] & \pi_0(\Omega^\infty KU) \cong \mathbb{Z}
\end{tikzcd}$$

$$GL_1(KU) \simeq (BU \times \mathbb{Z}/2\mathbb{Z})_\otimes$$

Theorem (Segal, May, Ando-Blumberg-Hopkins-Gepner-Rezk)

Let E be an E_∞-ring spectrum. Then there exists a spectrum $gl_1(E)$ such that

$$\Omega^\infty gl_1(E) \simeq GL_1(E)$$

$BGL_1(KU)$ and twists of K-theory

Interpretation of $gl_1(KU)^k(X)$ for $k \in \{0,1\}$

- $gl_1(KU)^0(X) = [X, GL_1(KU)] \xleftrightarrow{1:1}$ virtual line bundles$/X$
- $gl_1(KU)^1(X) = [X, BGL_1(KU)]$ classifies "bundles of free KU-modules of rank 1 over X", known as the twists of K-theory

Any hermitian line bundle over X gives an invertible element in $K^0(X)$.

The map $BU(1) \to GL_1(KU)$ deloops to

$$BBU(1) \simeq K(\mathbb{Z}, 3) \to BGL_1(KU) \ .$$

Twists that factor through this map are called geometric twists.

Definition (Toms-Winter)

A separable, unital C^*-algebra D is called strongly self-absorbing if $\exists$ an isomorphism $\psi\colon D \to D \otimes D$ and a sequence $(u_n)_{n\in\mathbb{N}}$ of elements in $U(D \otimes D)$ with

$$\lim_{n\to\infty} \|\psi(d) - u_n(d \otimes 1_D)u_n^*\| = 0 \ .$$

Strongly self-absorbing C^*-algebras in the UCT-class:

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[r] \arrow[dd] & M_P^{\otimes\infty} \arrow[r] \arrow[dd] & \mathcal{Q} \arrow[dr] \arrow[dd] & \\
\mathbb{C} \arrow[ur] \arrow[dr] & & & & \mathcal{O}_2 \\
 & \mathcal{O}_\infty \arrow[r] & \mathcal{O}_\infty \otimes M_P^{\otimes\infty} \arrow[r] & \mathcal{O}_\infty \otimes \mathcal{Q} \arrow[ur] &
\end{tikzcd}$$

Topological K-theory and its units

KU^D-theory

Let D be a strongly self-absorbing C^*-algebra. The functor $X \mapsto K_*(C(X) \otimes D)$ is a multiplicative cohomology theory,

Theorem (Joachim, Dadarlat-P, Land-Nikolaus, Bunke)

The functor $X \mapsto K_*(C(X) \otimes D)$ is represented by an E_∞-ring spectrum KU^D (even a commutative symmetric ring spectrum).

Examples

- $KU^{\mathcal{Z}} \simeq KU^{\mathcal{O}_\infty} \simeq KU^{\mathbb{C}} \simeq KU$
- $KU^{M_P^{\otimes\infty}} \simeq KU^{M_P^{\otimes\infty} \otimes \mathcal{O}_\infty} \simeq KU[P^{-1}]$

Dixmier-Douady theory for strongly self-abs. algebras

Definition

Let B be a C^*-algebra. A bundle of C^*-algebras $\mathcal{A} \to X$ with fibre B is a fibre bundle with structure group $\mathrm{Aut}(B)$. Let $\mathcal{B}un_B(X)$ be the isomorphism classes.

Theorem (Dixmier-Douady)

$$\mathcal{B}un_{\mathbb{K}}(X) \cong [X, BPU(H)] \cong H^3(X, \mathbb{Z}) \ .$$

Let D be a strongly self-absorbing C^*-algebra.

Theorem (Dadarlat-P.)

$\mathcal{B}un_{D\otimes\mathbb{K}}(X)$ is a group with respect to $\otimes_X$. If $D \neq \mathbb{C}$, then

$$\mathcal{B}un_{D\otimes\mathbb{K}}(X) \cong [X, B\mathrm{Aut}(D \otimes \mathbb{K})] \cong [X, BGL_1(KU^D)_+] \ .$$

How strongly self-absorbing helps

Theorem (Dadarlat-Winter)

D is strongly self-absorbing iff there exists an isomorphism $\psi\colon D \to D \otimes D$ and a path $u\colon [0,1) \to U(D \otimes D)$ such that

$$\lim_{t\to 1}\|\psi(d) - u(t)(d\otimes 1)u(t)^*\| = 0\ .$$

Lemma (Dadarlat-P.)

There is a continuous path $\psi\colon [0,1] \to \mathrm{hom}(D, D^{\otimes 2})$ such that

1. $\psi_0(d) = d \otimes 1 \quad , \quad \psi_1(d) = 1 \otimes d$
2. $Im(\psi|_{(0,1)}) \subset iso(D, D^{\otimes 2})$

How strongly self-absorbing helps (continued)

Define $H\colon \operatorname{Aut}(D) \times [0,1] \to \operatorname{Aut}(D)$ by

$$H(\alpha, t) = \begin{cases} \alpha & \text{for } t = 0\,, \\ \psi_t^{-1} \circ (\alpha \otimes \mathrm{id}) \circ \psi_t & \text{for } 0 < t < 1\,, \\ \mathrm{id} & \text{for } t = 1\,, \end{cases}$$

Theorem (Dadarlat-P.)

1. $\operatorname{Aut}(D)$ *is contractible.*
2. $\operatorname{Aut}_{1\otimes e}(D \otimes \mathbb{K})$ *is contractible.*
3. $\operatorname{Aut}_0(D \otimes \mathbb{K}) \simeq BU(D)$.

An ∞-categorical perspective

Definition (Lurie)

An idempotent object x in a symmetric monoidal ∞-category $\mathcal{C}$ with tensor unit $\mathbf{1}$ is a morphism $\varepsilon\colon \mathbf{1} \to x$ such that $\varepsilon \otimes \mathrm{id}_x \colon x \simeq \mathbf{1} \otimes x \to x \otimes x$ is an equivalence.

Let $C^*\mathrm{Alg}_h$ be the ∞-category obtained as the localisation of the category of C^*-algebras at the homotopy equivalences.

Examples

1. Let D be strongly self-absorbing.
   The unit map $\varepsilon\colon \mathbb{C} \to D$ is an idempotent object in $C^*\mathrm{Alg}_h$.
2. Let $e \in \mathbb{K}$ be a rank 1 projection.
   The map $\varepsilon\colon \mathbb{C} \to \mathbb{K}$ with $\varepsilon(1) = e$ is an idempotent in $C^*\mathrm{Alg}_h$

An ∞-categorical perspective (continued)

Let $\mathcal{C}$ be a stably symmetric monoidal ∞-category with unit $\mathbf{1}$. There are two algebra structures on $\mathrm{map}_{\mathcal{C}}(\mathbf{1},\mathbf{1})$.

Lemma (Bunke-Land-P.)

1. *The two algebra structures on* $\mathrm{map}_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ *agree.*
2. $\mathrm{map}_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ *is a commutative algebra in Sp.*
3. *If* $\varepsilon\colon \mathbf{1} \to x$ *is an idempotent, then* $L_x := \cdot \otimes x \colon \mathcal{C} \to \mathcal{C}$ *is a localisation functor.*
4. $\mathrm{map}_{\mathcal{C}}(x,x)$ *is a commutative algebra in Sp.*

Let D be strongly self-absorbing. Apply to $\mathcal{C} = \mathcal{KK}$ and $x = D$ to get

$$\mathrm{End}(D \otimes \mathbb{K}) \longrightarrow \Omega^\infty \mathcal{KK}(D,D) \xrightarrow{\simeq} \Omega^\infty \mathcal{KK}(\mathbb{C},D) \simeq \Omega^\infty KU^D \,.$$

and therefore also $\mathrm{Aut}(D \otimes \mathbb{K}) \to GL_1(KU^D)_+$.

Let G be a countable discrete group and let A be a unital C^*-algebra. An action of G on A is a homomorphism

$$G \to \operatorname{Aut}(A)\ .$$

Consider the following related notion:

Definition

A *G-kernel* (or *anomalous action*) is a group homomorphism

$$\bar{\alpha}\colon G \to \operatorname{Out}(A)\ ,$$

where $\operatorname{Out}(A) = \operatorname{Aut}(A)/\operatorname{Inn}(A)$ is the group of outer automorphisms.

Cocycle actions

Definition

A *cocycle action* (α, u) of G on A consists of two maps

$$\alpha \colon G \to \mathrm{Aut}(A) \quad , \quad u \colon G \times G \to U(A) ,$$

such that for all $g, h, k \in G$

$$\begin{aligned} \alpha_g \circ \alpha_h &= \mathrm{Ad}(u_{g,h}) \circ \alpha_{gh} , \\ \alpha_g(u_{h,k})u_{g,hk} &= u_{g,h}u_{gh,k} . \end{aligned}$$

- Second equation can be derived from associativity, compare

$$(\alpha_g \circ \alpha_h) \circ \alpha_k \quad \text{and} \quad \alpha_g \circ (\alpha_h \circ \alpha_k) .$$

The lifting obstruction

Assume $Z(A) = \mathbb{C}$. Let $\bar{\alpha}\colon G \to \mathrm{Out}(A)$ be a G-kernel.

- Lift $\bar{\alpha}$ to a map $\alpha\colon G \to \mathrm{Aut}(A)$.
- For $g, h \in G$ we have unitaries $u_{g,h} \in U(A)$ such that

$$\alpha_g \circ \alpha_h = \mathrm{Ad}(u_{g,h}) \circ \alpha_{gh} \ .$$

- For every triple $g, h, k \in G$ we have $\omega_{g,h,k} \in \mathbb{T}$ such that

$$\alpha_g(u_{h,k})u_{g,hk} = \omega_{g,h,k}\, u_{g,h}u_{gh,k} \ .$$

- The map $\omega\colon G \times G \times G \to \mathbb{T}$ is a cocycle representing an element

$$\mathrm{ob}(\bar{\alpha}) = [\omega] \in H^3(G, \mathbb{T}) \ .$$

Lemma

The class $\mathrm{ob}(\bar{\alpha})$ does not depend on the choice of unitaries. It vanishes if and only if $\bar{\alpha}$ lifts to a cocycle action (α, u).

G-kernels on the hyperfinite II_1-factor

Theorem (Connes, Jones, Ocneanu)

Let $A = R$ be the hyperfinite II_1-factor and let G be a discrete amenable group.

1. *The obstruction*
$$\mathrm{ob}(\bar{\alpha}) \in H^3(G, \mathbb{T})$$
is a complete invariant for injective G-kernels up to conjugacy.
2. *For every class $[\omega] \in H^3(G, \mathbb{T})$ there is an injective G-kernel $\bar{\alpha} \colon G \to \mathrm{Out}(R)$ such that $\mathrm{ob}(\bar{\alpha}) = [\omega]$.*

Group actions and classifying spaces

Theorem (Izumi, Meyer, Gabe-Szabo)

- *D strongly self-absorbing Kirchberg algebra*
- *G countable discrete amenable group*
- *BG has a finite CW-complex model*

$$\{\textit{outer cocycle actions of } G \textit{ on } D\}/\sim_{cc} \xleftrightarrow{1:1} [BG, B\mathrm{Aut}_0(D \otimes \mathbb{K})]$$

Theorem (Dadarlat-P.)

$[BG, B\mathrm{Aut}_0(D \otimes \mathbb{K})]$ is group with respect to $\otimes_X$. It is the first group of a cohomology theory $\bar{E}^*_D(\,\cdot\,)$ with

$$\bar{E}^1_D(BG) = [BG, B\mathrm{Aut}_0(D \otimes \mathbb{K})] \cong [BG, BSL_1(KU^D)]\ .$$

Equivariant automorphisms and stable homotopy theory

Theorem (Kamikawa, Evans-P.)

- *D unital separable C^*-algebra*
- *G compact group*

Let $\gamma \colon G \curvearrowright D$ be a strongly self-absorbing action such that

1. *D^γ is K_1-injective,*
2. *$(D \otimes \mathbb{K})^{\gamma \otimes \mathrm{id}_{\mathbb{K}}}$ has the cancellation property.*

Then $\mathrm{Aut}_G(D \otimes \mathbb{K})$ is an infinite loop space, such that

$$E^0_{G,D}(X) = [X, \mathrm{Aut}_G(D \otimes \mathbb{K})] \quad , \quad E^1_{G,D}(X) = [X, B\mathrm{Aut}_G(D \otimes \mathbb{K})] \ .$$

Automorphisms and equivariant stable homotopy theory

Let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime p. Let V be a unitary G-representation with $\dim(V) < \infty$, which contains at least two non-isomorphic irreps and

$$D = \mathrm{End}(V)^{\otimes \infty}$$

Let $\mathbb{K}$ be the compact operators on a full G-universe.

Theorem (Bianchi-P.)

The group $\mathrm{Aut}(D \otimes \mathbb{K})$ is a G-equivariant infinite loop space. The associated G-equivariant cohomology theory $E_D^*(X)$ satisfies

$$E_D^*(X) \cong gl_1(KU^D)_+^*(X)$$

and the first group classifies equivariant bundles over finite G-CW-complexes.

Consider the top. 2-category $\mathcal{G}_A$ (top. 2-group)

- a single object (denoted by $*$),
- 1-morphisms given by $\mathrm{Aut}(A)$,
- 2-morphisms given by $\mathrm{Aut}(A) \times U(A)$,
  where $(\alpha, u) \in \mathrm{Aut}(A) \times U(A)$ gives a 2-morphism $\alpha \Rightarrow \mathrm{Ad}(u) \circ \alpha$.

$$\begin{tikzcd}[column sep=huge]
* & * \arrow[l, bend right=70, "\alpha"'] \arrow[l] \arrow[l, bend left=70, "\mathrm{Ad}(u_1u_2)\circ\alpha"] \arrow[l, phantom, shift right=4ex, "\Downarrow u_2"] \arrow[l, phantom, shift left=4ex, "\Downarrow u_1"]
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}[column sep=huge]
* & * \arrow[l, bend right=70, "\alpha"'] \arrow[l, bend left=70, "\mathrm{Ad}(u_1u_2)\circ\alpha"] \arrow[l, phantom, "\Downarrow u_1u_2"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
* & * \arrow[l, bend right=70, "\alpha"'] \arrow[l, bend left=70, "\mathrm{Ad}(u)\alpha"] \arrow[l, phantom, "\Downarrow u"] & * \arrow[l, bend right=70, "\beta"'] \arrow[l, bend left=70, "\mathrm{Ad}(v)\beta"] \arrow[l, phantom, "\Downarrow v"]
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}[column sep=huge]
* & * \arrow[l, bend right=70, "\alpha\beta"'] \arrow[l, bend left=70, "\mathrm{Ad}(u\alpha(v)))\,\alpha\beta"] \arrow[l, phantom, "\Downarrow u\alpha(v)"]
\end{tikzcd}$$

A cocycle action is a pseudofunctor $(\alpha, u)\colon \mathcal{C}_G \to \mathcal{G}_A$

$$
\begin{tikzcd}
* & & * \arrow[ll, "\alpha_{gh}"'] \arrow[dl, "\alpha_h"] \\
& * \arrow[ul, "\alpha_g"] \arrow[u, phantom, "\Downarrow u_{g,h}"] &
\end{tikzcd}
\qquad\qquad
\mathrm{Ad}(u_{g,h})\,\alpha_{gh} = \alpha_g\,\alpha_h
$$

where the following tetrahedron gives the associativity constraint:

$$
\begin{tikzcd}
& * \arrow[dl, "\alpha_g"'] \arrow[d, phantom, "\Uparrow u_{g,h}"] & & \\
* & {} & * \arrow[ll, "\alpha_{gh}"] \arrow[ul, "\alpha_h"'] \arrow[d, phantom, "\Uparrow u_{gh,k}"] & * \arrow[l, "\alpha_k"'] \arrow[lll, bend left=50, "\alpha_{ghk}"] \\
& & {} &
\end{tikzcd}
=
\begin{tikzcd}
& & * \arrow[dl, "\alpha_h"'] \arrow[d, phantom, "\Uparrow u_{h,k}"] & \\
* & * \arrow[l, "\alpha_g"'] & {} \arrow[d, phantom, "\Uparrow u_{g,hk}"] & * \arrow[ll, "\alpha_{hk}"] \arrow[ul, "\alpha_k"'] \arrow[lll, bend left=50, "\alpha_{ghk}"] \\
& & {} &
\end{tikzcd}
$$

Topological crossed modules

Definition

A *(topological) crossed module* $\mathcal{G} = (\Gamma_0, \Gamma_1, \partial)$ is

- a homorphism $\partial\colon \Gamma_1 \to \Gamma_0$ of top. groups,
- a continuous left action of Γ_0 on Γ_1 (denoted by $\alpha(u)$),

such that for $\alpha \in \Gamma_0$ and $u, v \in \Gamma_1$

$$\partial(\alpha(u)) = \alpha\,\partial(u)\,\alpha^{-1} \quad , \quad \partial(u)(v) = u\,v\,u^{-1}\,.$$

$\mathcal{G}$ gives rise to a 2-category with a single object $*$ and

- 1-morphisms given by Γ_0,
- 2-morphisms given by $\Gamma_0 \times \Gamma_1$ with $(\alpha, u)\colon \alpha \Rightarrow \partial(u)\alpha$.

Crossed modules associated to C^*-algebras

Let A be a unital C^*-algebra.

Examples

1. $\mathcal{G}_A = (\mathrm{Aut}(A), U(A), \mathrm{Ad})$ with $\mathrm{Ad}(u)(a) = uau^*$ and the canonical action of $\mathrm{Aut}(A)$ on $U(A)$.

$$H^1(G, \mathcal{G}_A) \xleftrightarrow{1:1} \{\text{cocycle actions on } A\}/\sim_{cc}$$

2. $P\mathcal{G}_A = (\mathrm{Aut}(A), PU(A), \mathrm{Ad})$ with $PU(A) = U(A)/Z(U(A))$.

$$H^1(G, P\mathcal{G}_A) \xleftrightarrow{1:1} \{G\text{-kernels on } A\}/\sim_c$$

3. If $U(A)$ is connected, then $\widetilde{\mathcal{G}}_A = (\mathrm{Aut}(A), \widetilde{U(A)}, \widetilde{\mathrm{Ad}})$ where $\widetilde{U(A)}$ is the universal cover of $U(A)$.

Lifting obstructions and short exact sequences

Let A be a unital C^*-algebra with $Z(A) = \mathbb{C}$ and let $\mathcal{Z}_A = (\{1\}, \mathbb{T}, \mathrm{triv})$.

The short exact sequence of crossed modules

$$\begin{tikzcd}
\mathbb{T} \arrow[r] \arrow[d] & U(A) \arrow[r] \arrow[d, "\mathrm{Ad}"] & PU(A) \arrow[d, "\mathrm{Ad}"] \\
1 \arrow[r] & \mathrm{Aut}(A) \arrow[r, equal] & \mathrm{Aut}(A)
\end{tikzcd}$$

gives rise to a long(-ish) exact sequence of pointed sets

$$\dots \longrightarrow H^1(G, \mathcal{Z}_A) \longrightarrow H^1(G, \mathcal{G}_A) \longrightarrow H^1(G, P\mathcal{G}_A) \xrightarrow{\mathrm{ob}} H^2(G, \mathcal{Z}_A)$$

Note that $H^2(G, \mathcal{Z}_A)$ only exists, because $\mathbb{T}$ is abelian. In fact,

$$H^2(G, \mathcal{Z}_A) \cong H^3(G, \mathbb{T}) \, .$$

Mapping algebra to homotopy theory

Definition

The *classifying space* $B^D\mathcal{G}$ is the geometric realisation of the Duskin nerve $N^D_*(\mathcal{G})$.

Each pseudofunctor $(\alpha, u)\colon \mathcal{C}_G \to \mathcal{G}$ gives a continuous map

$$BG \to B^D\mathcal{G} \ .$$

Lemma (Girón Pacheco, Izumi, P.)

This provides a well-defined natural transformation of pointed sets

$$H^1(G, \mathcal{G}) \to [BG, B^D\mathcal{G}] \ .$$

Classification via cohomology theories

Theorem (Girón Pacheco, Izumi, P.)

Let A be unital separable C^*-algebra. There is a weak homotopy equivalence

$$B^D\mathcal{G}_A \simeq B\mathrm{Aut}_0(A \otimes \mathbb{K}) \ .$$

Theorem (P.-Tridimas)

Let A be a strongly self-absorbing C^*-algebra. The spaces $B^D\mathcal{G}_A$, $B^DP\mathcal{G}_A$, $B^D\widetilde{\mathcal{G}}_A$ all have natural infinite loop space structures.

Consequently, the natural transformation

$$H^1(G, \mathcal{G}) \to [BG, B^D\mathcal{G}]$$

takes values in abelian groups for $\mathcal{G} \in \{\mathcal{G}_A, P\mathcal{G}_A, \widetilde{\mathcal{G}}_A\}$.

The End

Thank you!