C^*-algebras

Def.: An algebra A over $\mathbb{C}$ together with an anti-linear involution $a \mapsto a^*$ s.th. $(ab)^* = b^* a^*$ and equipped with a norm $a \mapsto \|a\|$ is called a C^*-algebra if

i) A is a Banach space w.r.t. $\|\cdot\|$ and $\|ab\| \leq \|a\| \cdot \|b\|$,

ii) ~~Haa~~ $\|a^* a\| = \|a\|^2 \quad \forall a \in A$.

Examples:

- $\mathbb{C}$ with $|\cdot|$ and $z^* = \bar{z}$
- H Hilbert space, then $B(H)$ with
$$\|T\| = \sup_{\|\xi\| \leq 1} \|T\xi\| \quad \text{and } T^* \text{ the adjoint of } T$$
is a C^*-algebra. In particular $M_n(\mathbb{C}) = B(\mathbb{C}^n)$
- H separable infinite-dim. Hilbert space
e_i Hilbert basis of H
induces inclusions $M_n(\mathbb{C}) \longrightarrow B(H)$
s.th.
$$\begin{tikzcd} M_n(\mathbb{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & B(H) \\ M_{n+1}(\mathbb{C}) \arrow[ur] & \end{tikzcd} \quad \rightsquigarrow \quad \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C}) \hookrightarrow B(H)$$
$$\mathbb{K}(H) = \overline{\bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})}^{\|\cdot\|} \text{ in } B(H)$$
$\uparrow$ ~~cpt~~ compact operators

Non-example: $\ell^1(\Gamma)$ for a discrete group Γ

- X locally compact ~~space~~, Hausdorff space
$C_0(X)$ continuous functions on X s.th.
$\forall \varepsilon > 0 \; \exists K \subset X$ cpt. with $|f(x)| < \varepsilon \; \forall x \in X \setminus K$.
with $f^*(x) = \overline{f(x)}$ and $\|f\| = \sup_{x \in X} |f(x)|$

Thm (Gelfand–Naimark): Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally cpt. ~~space~~ Hausdorff space X.

Idea of the proof: ~~X = Chara~~ A comm. C^*-alg.

$X = \hom(A, \mathbb{C})$ equipped with the weakest top. s.th. $\chi \mapsto \chi(a)$ is cont.

$$A \xrightarrow{\cong} C_0(X)$$
$$a \longmapsto (\chi \mapsto \chi(a)) \,..$$

Slogan: Theory of C^*-algebras is "noncommutative topology".

K-Theory of C^*-algebras

A unital C^*-algebra.

Def.: $p \in A$ is called a projection if $p = p^2 = p^*$. $P(A)$ = set of projections in A.

$$P_n(A) = P(M_n(A)) \quad \text{and} \quad P_\infty(A) = \coprod_{n=1}^{\infty} P_n(A)$$

$p \in P_n(A)$, $q \in P_m(A)$, then $p \sim q$ iff $\exists\, v \in M_{m,n}(A)$ s.th.

$$v^*v = p \quad \text{and} \quad vv^* = q.$$

$D(A) = P_\infty(A)/_\sim$ This is a abelian semigroup: $p \in P_n(A)$, $q \in P_m(A)$

$$[p] \oplus [q] := \Big[\underbrace{\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}}_{\in P_{m+n}(A)}\Big].$$

$$K_0(A) = Gr(D(A))$$

Grothendieck group, i.e. pairs $([p],[q])$ considered as formal differences $[p]-[q]$.

Ex.: $A = \mathbb{C}$. Then $[p] = [q]$ in $D(A)$ iff $tr(p) = tr(q)$.

$$\Rightarrow D(\mathbb{C}) = \mathbb{N} \Rightarrow K_0(\mathbb{C}) = \mathbb{Z}.$$

Def.: $u \in A$ is called a unitary if $uu^* = u^*u = 1$. $U(A)$ = group of unitary elements in A

$$U_n(A) = U(M_n(A)) \quad \text{and} \quad U_\infty(A) = \coprod_{n=1}^{\infty} U_n(A)$$

$u \in U_n(A)$, $v \in U_m(A)$, then $u \sim v$ if $\exists\, K \in \mathbb{N}$, $K \geq n$, $K \geq m$ s.th.

$\begin{pmatrix} u & 0 \\ 0 & 1_{K-n} \end{pmatrix}$ is ~~homotopic~~ connected to $\begin{pmatrix} v & 0 \\ 0 & 1_{K-m} \end{pmatrix}$ by a path in $U_K(A)$.

$U_\infty(A)/_\sim$ is an abelian semigroup w.r.t. $[u] \oplus [v] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]$

Lemma: $U_\infty(A)/_\sim$ is actually a group with $-[u] = [u^*]$

Sketch of proof: If $u \in U_m(A)$ and $v \in U_m(A)$, then

$$\left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right] = \left[\begin{pmatrix} uv & 0 \\ 0 & 1_m \end{pmatrix}\right] \quad \text{using rotation matrices.}$$

$$K_1(A) = U_\infty(A)/_\sim.$$

Ex.: $K_1(\mathbb{C}) = 0$ since $U_n(\mathbb{C}) = U(n)$ is path-connected.

in the non-unital case: Replace by unitization $A^+ = A \oplus \mathbb{C}$

and define $K_0(A) := \mathrm{Ker}\left(K_0(A^+) \longrightarrow K_0(\mathbb{C})\right)$

$K_1(A) := K_1(A^+)$.

<u>Bott periodicity</u>

Given a C^*-algebra A, define the suspension of A to be

$$SA = \{ f \in C([0,1], A) \mid f(0) = f(1) = 0 \}$$

<u>Thm.</u>: $K_1(A) \cong K_0(SA)$

This gives an idea how to define higher K-groups inductively

$$K_n(A) := K_{n-1}(SA).$$

<u>Thm</u> (Bott periodicity): $K_0(A) \cong K_1(SA) = K_2(A)$, i.e.

$K_n(A) \cong K_{n+2}(A) \quad \forall n \in \mathbb{N}_0$.

$f_p(z) = zp + (1_n - p)$

<u>Relation with topological K-theory</u>

X cpt. Hausdorff space

<u>Def.</u>: $K^0(X)$ = Grothendieck group of iso-classes of fin. dim. cplx. vector bdl's over X

$K^1(X) = GL(C(X)) / GL(C(X))_0$

<u>Thm</u> (Serre-Swan): $K^0(X) \cong K_0(C(X))$

<u>Idea of the proof</u>: .) Given a vector bdl. $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ there is a vector bdl. $\begin{matrix} F \\ \downarrow \\ X \end{matrix}$

s.th. $E \oplus F \cong X \times \mathbb{C}^n$ as vector bdl.'s.

.) Can ~~ensure~~ w.l.o.g. assume that F is the <u>orth.</u> complement of E in $X \times \mathbb{C}^n$ w.r.t. the standard ~~scala~~ inner product.

$E \cong \{ (x, v) \in X \times \mathbb{C}^n \mid p(x)v = v \}$ for some projection valued function $p : X \longrightarrow M_n(\mathbb{C})$

$$\begin{aligned} K^0(X) &\longrightarrow K_0(C(X)) \\ [E] &\longmapsto [p] \end{aligned} \quad \Big| \quad p \in M_n(C(X))$$

K-Theory from the topological viewpoint

· Def.: Let H be a sep. inf. dim. Hilbert space

Define $BU(n) = \{ n\text{-dim subspaces of } H \}$.

Bijection
$$U(H) / U(n) \times U((\mathbb{C}^n)^\perp) \longrightarrow BU(n)$$

which endows $BU(n)$ with a topology.

Vector bdl. $EU(n) \longrightarrow BU(n)$ given by

$$EU(n) = \{ (x, v) \in BU(n) \times H \mid v \in X \}$$

Thm.: $[\overset{\text{cpt. Hausdorff}}{X}, BU(n)] \cong$ iso-classes of n-dim. vector bdls. over X

via $[f] \longmapsto E = f^* EU(n)$.

·) $V \in BU(n)$ yields new subspace $V \oplus \mathbb{C}$ in $H \oplus \mathbb{C} \cong H$

induces a map $BU(n) \longrightarrow BU(n+1)$

Def.: $BU = \operatorname{colim}_n BU(n)$

Thm.: $K^0(X) = [X, BU \times \mathbb{Z}]$

Idea: Given $f : X \longrightarrow BU \times \mathbb{Z}$ it factors through some $BU(n) \times \mathbb{Z}$

yields
$$\begin{array}{c} E \leftarrow n\text{-dim. vector bdl.} \\ \downarrow \\ X \end{array}$$

$[f]$ is mapped to $[E] - [\mathbb{C}^k]$ where k is such that $n - k$ is the $\mathbb{Z}$-value of f.

$K^1(X) = [X, U_\infty]$

Thm. (Bott): $\Omega U_\infty \simeq BU \times \mathbb{Z}$