The Erdös-Szekeres Convex Polygon Problem

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The Convex Polygon Problem

Question (Esther Klein '33)

For any integer $n \geq 3$, what is the smallest positive integer $ES(n)$ such that any set of at least $ES(n)$ points in the plane in general position contains n points in convex position?

Definition

A finite point set $S \subset \mathbb{R}^2$ is in *general position* if no three points lie on a line. It is in *convex position*, if for all $p \in S$ we have

$$p \notin \operatorname{conv}(S \setminus \{p\})$$

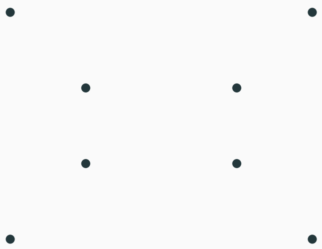
(equivalently: S consists of the vertices of a convex polygon.)

The Convex Polygon Problem

Question

How many points do I need at least until I am guaranteed to find a convex n-gon?

For example for $n = 5$:



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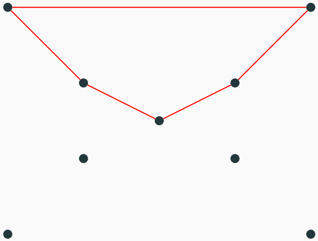
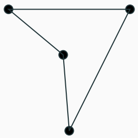
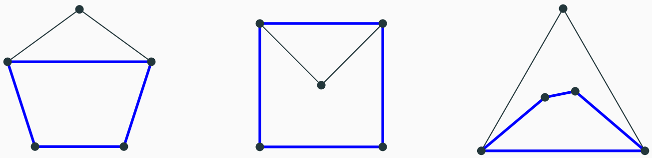


Klein's proof of $ES(4) = 5$

The following four point set has no four points in convex position proving $ES(4) > 4$:



Looking at the convex hull of five points, we can distinguish the following cases, each of which contains convex quadrilateral proving $ES(4) \leq 5$:



A Short History of the Problem

Exact solutions:

- Klein '33: $ES(4) = 5$
- Makai '35: $ES(5) = 9$
- Szekeres-Peters '06: $ES(6) = 17$

Conjecture (Erdös-Szekeres '35):

$$ES(n) = 2^{n-2} + 1$$

Theorem (Erdös-Szekeres '35)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1$$

The Happy Ending Problem

Figure 1: George and Esther Szekeres (at the Investiture Ceremony for Australian Honours in New South Wales).

A Short History of the Problem (contd.)

Upper Bounds:

- Chung-Graham '98: $ES(n) \leq \binom{2n-4}{n-2}$
- Kleitman-Pachter '98: $ES(n) \leq \binom{2n-4}{n-2} - 2n + 7$
- Tóth-Valtr '98: $ES(n) \leq \binom{2n-5}{n-2} + 2$
- Tóth-Valtr '05: $ES(n) \leq \binom{2n-5}{n-2} + 1$

Theorem (Norin-Yuditsky '15, Mojarrad-Vlachos '15)

$$\limsup_{n \to \infty} \frac{ES(n)}{\binom{2n-5}{n-2}} \leq \frac{7}{8}$$

The Cups-Caps Theorem

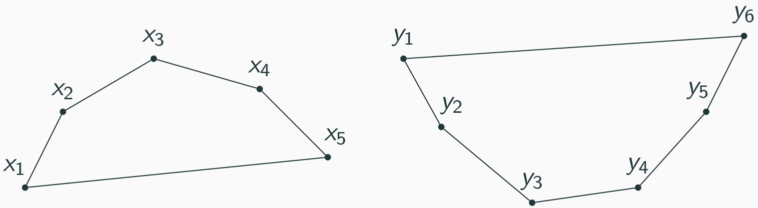


Figure 2: A 5-cap (left) and a 6-cup (right)

Theorem (Erdös-Szekeres '35)

Let $f(k,l)$ be the smallest integer with the property that any subset $P \subset \mathbb{R}^2$ with $f(k,l)$ points contains a k-cup or an l-cap. Then

$$f(k,l) = \binom{k+l-4}{k-2} + 1 \ .$$

Suk's Theorem

Theorem (Suk '16)

For all $n \geq n_0$, where n_0 is a sufficiently large constant, we have

$$ES(n) \leq 2^{n+4n^{4/5}} .$$

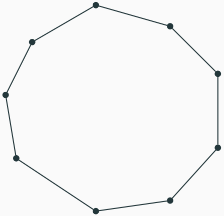


Figure 3: General convex set is a combination of a cup and a cap.

Transitive 2-Colourings

Let $S = \{1, \dots, N\}$ and denote by $\mathcal{P}_r(S)$ the set of r-element subsets of S.

Definition

A 2-*colouring of the triples* of S is a map $c\colon \mathcal{P}_3(S) \to \{0, 1\}$. Such a 2-colouring is called *transitive* if for $i < j < k < l$ such that $c(\{i, j, k\}) = c(\{j, k, l\})$ we have that

$$c(\{i, k, l\}) = c(\{i, j, l\}) = c(\{i, j, k\}) \ .$$

Theorem

Let $g(k, l)$ be the minimal integer N such that for every transitive 2-colouring of the triples of $S = \{1, \dots, N\}$ there exists

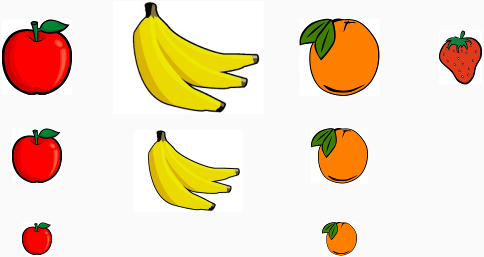
- *$X \subset S$ with $|X| = k$ and $c|_{\mathcal{P}_3(X)} \equiv 0$ or*
- *$Y \subset S$ with $|Y| = l$ and $c|_{\mathcal{P}_3(Y)} \equiv 1$.*

Then $g(k, l) = f(k, l) = \binom{k+l-4}{k-2} + 1$.

Dilworth's Theorem

Theorem

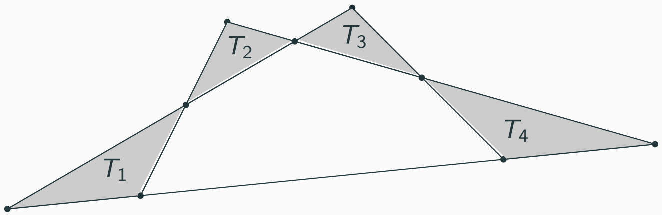
Let $(P, \preceq)$ be a finite poset. The minimal number of disjoint chains covering P is equal to the maximal size of an antichain in P.



Corollary

If $|P| \geq m \cdot n$, then it either contains a chain of size at least m or an antichain of size at least n.

The Partitioned Erdös-Szekeres Theorem



Theorem (Pór-Valtr '02)

Let $k \geq 3$ and let P be a finite point set in the plane in general position with $|P| \geq 4^k$. Then there is a k-element subset $X \subset P$ such that X is either a k-cup or a k-cap and the regions $T_1, \ldots, T_{k-1}$ from the support of X satisfy

$$|T_i \cap P| \geq \frac{|P|}{2^{40k}} \; .$$

The Proof - Part 1

Theorem (Suk '16)

For all $n \geq n_0$, where n_0 is a sufficiently large constant, we have

$$ES(n) \leq 2^{n+4n^{4/5}} \ .$$

Proof:

- $N = \lfloor 2^{n+4n^{4/5}} \rfloor$, P - point set in the plane with $|P| = N$,
- $k = \lceil \sqrt{n} \rceil$

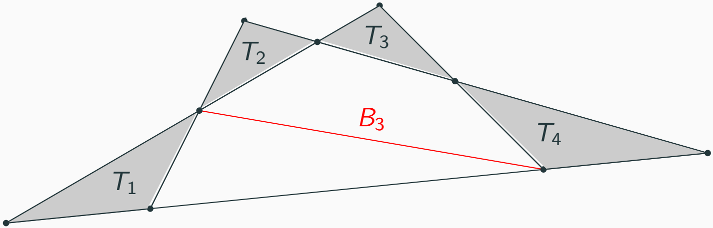
Apply the partitioned Erdös-Szekeres Theorem to P for $k+3$. We obtain a subset $X = \{x_1, \ldots, x_{k+3}\} \subset P$, which is a cup or a cap with support regions $T_1, \ldots, T_{k+2}$. Let $P_i = T_i \cap P$. Then

$$|P_i| \geq \frac{N}{2^{50k}} \ .$$

The Proof - Part 1

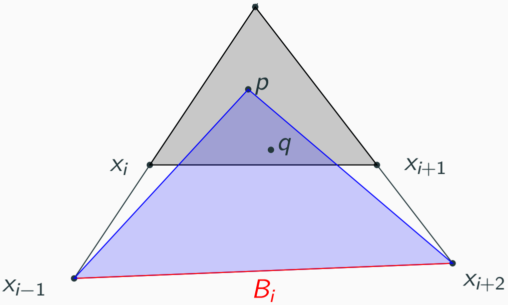
(We will assume that we have a cap. The cup-case is analogous.)

Let B_i be the segment from x_{i-1} to x_{i+2}.



and introduce a partial order $\prec$ in each set $P_i \subset T_i$ for $i \in \{2, \ldots, k+1\}$ as described on the next slide.

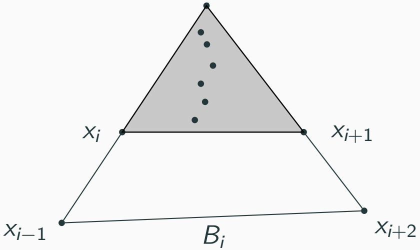
The Partial Order in T_i



Definition

$p \prec q$ if $p \neq q$ and $q \in \text{conv}(B_i \cup \{p\})$.

The Partial Order in T_i



Observation

If $p_1 \prec p_2 \prec \cdots \prec p_r$ is a chain in T_i with respect to $\prec$, then the line through any two p_j intersects B_i and therefore avoids the adjacent regions of the support.

If $\{q_1, \ldots, q_s\}$ is an antichain in T_i with respect to $\prec$, a line through two q_j's avoids all regions $T_1, \ldots, T_{i-2}$ and $T_{i+2}, \ldots, T_n$.

The Proof - Part 2

Theorem (Suk '16)

For all $n \geq n_0$, where n_0 is a sufficiently large constant, we have

$$ES(n) \leq 2^{n+4n^{4/5}} .$$

Proof (contd.):

- $\alpha = n^{-\frac{1}{5}}$.

Apply the corollary of Dilworth's theorem to each P_i to get that P_i contains

- a *chain* of length at least $|P_i|^{1-\alpha}$ or
- an *antichain* of size at least $|P_i|^{\alpha}$.

The Proof - The Antichain Case

Assume that at least $n^{1/4}$ of the P_i in $\{P_2, \ldots, P_{k+1}\}$ contain an antichain $Q_i \subset P_i$ such that $|Q_i| \geq |P_i|^\alpha$. Then there are at least

$$t = \left\lceil \frac{n^{1/4}}{2} \right\rceil$$

non-adjacent ones. Let these be $Q_{j_1}, \ldots, Q_{j_t}$.

For n_0 large enough...

$$|Q_{j_r}| \geq |P_{j_r}|^\alpha \geq \left(\frac{N}{2^{50k}}\right)^\alpha \geq f(n, \lceil 2n^{3/4} \rceil) \; .$$

If any of the Q_{j_r} contains an n-cup we are done. Therefore assume that each Q_{j_r} contains a $\lceil 2n^{3/4} \rceil$-cap $S_{j_r} \subseteq Q_{j_r}$.

The Proof - The Antichain Case

Observation

Let $S = S_{j_1} \cup \cdots \cup S_{j_t}$. We can find a line through two points in S_{j_r} such that any other point of S_{j_r} lies below it. Since this line avoids all other regions T_i, all other points of S lie below it. Hence S is a cap.

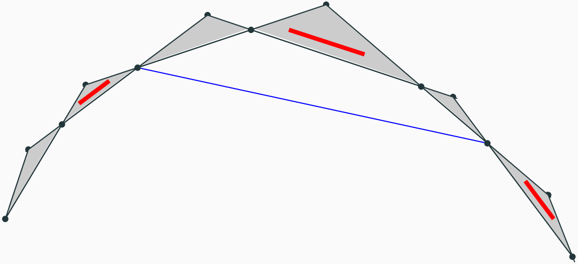


Figure 4: The antichain case: The caps S_{j_r} are sketched as red lines.

The Proof - The Antichain Case

But we have...

$$|S| \geq \frac{n^{1/4}}{2} \cdot 2n^{3/4} = n \ .$$

Hence, P contains a cap of size n, which concludes this case.

The Proof - The Chain Case

If we have less than $\lceil n^{1/4} \rceil$ of the P_i containing antichains, then each remaining P_j contains a chain Q_j of length at least $|P_j|^{1-\alpha}$.

By the pigeonhole principle and our choice of $k = \lceil n^{1/2} \rceil$ there are at least $t = \lceil n^{1/4} \rceil$ sets Q_j with consecutive indices j.

Rename those chains to $Q_1, \ldots, Q_t$ and order the elements in each Q_j with respect to $\prec$.

Definition

A subset $Y \subset Q_i$ is a *right cap* if $\{x_i\} \cup Y$ is in convex position and it is a *left cap* if $\{x_{i+1}\} \cup Y$ is in convex position.

The Proof - The Chain Case

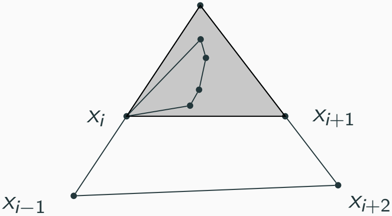


Figure 5: A right cap in the support region T_i.

Colour the triples in each Q_i with colour 0 if the triple is a left cap and with colour 1 if it forms a right cap.

Observations

- Q_i is a chain $\Rightarrow$ every triple is either a left or a right cap, not both.
- If $(q_{i_1}, q_{i_2}, q_{i_3})$ and $(q_{i_2}, q_{i_3}, q_{i_4})$ are left (resp. right) caps, then $(q_{i_1}, q_{i_2}, q_{i_4})$ and $(q_{i_1}, q_{i_3}, q_{i_4})$ are left (resp. right) caps as well.

  $\Rightarrow$ The colouring is transitive.

The Proof - The Chain Case

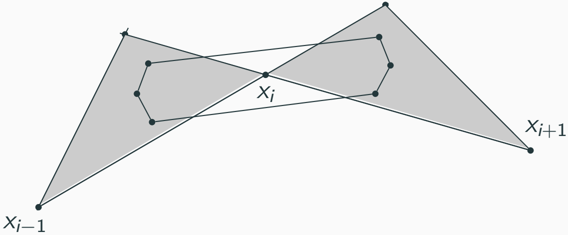


Figure 6: A left k-cap in T_{i-1} combines with a right ℓ-cap in T_i to form a $(k+\ell)$-convex set.

Observation

Suppose Q_{i-1} contains a left k-cap S_{i-1} and Q_i contains a right ℓ-cap S_i. Then $S_{i-1} \cup S_i$ is a set of $k+\ell$ points in convex position.

The Proof - The Chain Case

For n_0 large enough...

Let $K = \lceil n^{3/4} \rceil$.

$$|Q_i| \geq |P_i|^{1-\alpha} \geq \left(\frac{N}{2^{50k}}\right)^{1-\alpha} \geq \binom{n+K-4}{iK-2} + 1 = f(iK, n - iK + K)$$

Apply the theorem about transitive 2-colourings to each Q_i.

- Q_1 either contains a right n-cap or a left K-cap. Assume latter.
- Q_2 either contains a right $(n-K)$-cap or a left $2K$-cap.
- Right $(n-K)$-cap in Q_2 combines with left K-cap in Q_1.

The Proof - The Chain Case

- Q_i either contains a right $(n-(i-1)K)$-cap or a left iK-cap.
- Right $(n-(i-1)K)$-cap in Q_i combines with the left $(i-1)K$-cap in Q_{i-1} and we are done. So assume we have a left iK-cap.
- Proceed this way until Q_t for $t = \lceil n^{1/4} \rceil$, which then contains a left cap of size

$$tK = \lceil n^{1/4} \rceil \lceil n^{3/4} \rceil \geq n \ .$$

$\square$

Thank you!