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Higher twisted K-theory

The Interplay between Operator Algebras and Homotopy Theory in K-theory

vorgelegt von

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1 Introduction

This introduction starts with an overview about classical twisted K-theory and an outline of some of its applications, where we restrict our attention to the ones that were motivated by mathematical physics. For the reader who is just interested in the main results and has some background in twisted K-theory we recommend to go straight to section 1.3 or 1.4, which describe the goals of the project underlying this thesis and contain summaries of the papers that came out of it. In particular, section 1.4 is meant as a reading guide for the following chapters.

1.1 Twisted K-theory

Twisted K-theory was first introduced by Donovan and Karoubi in [56], where it was called "K-theory with local coefficients". It is a functor from a category of algebra bundles over spaces to graded abelian groups. More precisely: Let X be a locally compact, paracompact Hausdorff space and let $\mathcal{A} \to X$ be a locally trivial bundle of simple finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded complex associative unital algebras. The twisted K-group $K^n_{\mathcal{A}}(X)$ is defined to be the nth K-group obtained from the Banach category of graded finitely generated projective modules over the section algebra $C_0(X, \mathcal{A})$, i.e. $K^n_{\mathcal{A}}(X) = K_n(C_0(X, \mathcal{A}))$. The local triviality of $\mathcal{A} \to X$ implies that the groups are locally indistinguishable from ordinary K-theory, which yields a justification for the name twisted K-theory.

Twisted K-theory shares some similarities with the local coefficient version of ordinary cohomology in that it allows to circumvent orientability conditions. For example, there is a twisted version of the Thom isomorphism, which takes the following form: Let M be a smooth manifold and let $V \to M$ be an oriented real vector bundle equipped with a positive definite quadratic form. Let r be the rank of V. Then there is an isomorphism [82, Thm. 4.1]

$$K^n_{\mathbb{C}\ell(V)}(M) \to K^n(V) \ .$$

The left hand side of this map is the nth twisted K-group of M with the twist given by the Clifford bundle of V. If V is K-orientable, which is the case if and only if V carries a spin^c-structure, a choice of such an orientation induces an isomorphism $K^n_{\mathbb{C}\ell(V)}(M) \cong K^{n+r}(M)$. This is most easily seen for vector bundles of even rank, since then the choice of a spin^c-

structure yields a Morita equivalence between $C(M, \mathbb{C}\ell(V))$ and $C(M)$. In both cases the above isomorphism boils down to the ordinary Thom isomorphism $K^n(M) \to K^{n+r}(V)$.

The collection of all bundles $\pi\colon \mathcal{A} \to X$ as above yields objects in a category $\mathcal{B}$, in which the morphisms are given by bundle maps that are fiberwise algebra homomorphisms. They fit into commutative diagrams of the form

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{A}' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow[\bar{\varphi}]{} & X' \end{array}$$

Twisted K-theory defines a functor $K\colon \mathcal{B} \to \mathrm{GrAb}$ to the category of $\mathbb{Z}$-graded abelian groups. For this functor all of the properties of a generalized cohomology theory still hold, i.e. it is homotopy invariant, there is a relative version $K^n_{\mathcal{A}}(X, Y)$ for pairs of spaces $X \supset Y$ and corresponding long exact sequences, which boil down to 6-term exact sequences via Bott periodicity. There is a corresponding Mayer-Vietoris sequence and $K^*_{\mathcal{A}}(X)$ is a module over ordinary K-theory in a natural way.

The property of being a module can be extended to a multiplicative structure on twisted K-theory reflecting the fact that the above twists can be organized into a group. In fact, the family of algebras considered above is closed with respect to forming tensor products. Thus, if $\mathcal{A} \to X$ and $\mathcal{A}' \to X$ are two algebra bundles representing twists, their fiberwise graded tensor product $\mathcal{A} \otimes \mathcal{A}' \to X$ yields another one. Moreover, if $\mathcal{A}$ is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism bundle $\mathrm{End}(E) \to X$ of a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E \to X$, the space of continuous sections $C_0(X, E)$ provides a Morita equivalence between $C_0(X)$ and $C_0(X, \mathcal{A})$. In particular, $K^*_{\mathcal{A}}(X)$ is in this case isomorphic to $K^*(X)$. A crucial caveat in this observation is that the isomorphism is *non-canonical.* It depends on the choice of E. Motivated by these facts, we define the following equivalence relation: Call two algebra bundles $\mathcal{A}$ and $\mathcal{A}'$ over X equivalent if there exist graded vector bundles E and E' and an isomorphism

$$\mathcal{A} \otimes \mathrm{End}(E) \cong \mathcal{A}' \otimes \mathrm{End}(E') \ . \tag{1.1}$$

The equivalence classes of algebra bundles form a group classifying these twists of K-theory. This group is called the *graded Brauer group* $\mathrm{GBr}(X)$ in [56, Sec. 2 and 3]. It was proven in [56, Thm. 11] that for a finite CW-complex X there is a natural isomorphism

$$\mathrm{GBr}(X) \cong H^0(X, \mathbb{Z}/2\mathbb{Z}) \times H^1(X, \mathbb{Z}/2\mathbb{Z}) \times \mathrm{Tor}(H^3(X, \mathbb{Z})) \tag{1.2}$$

where the group structure on the right hand side is given by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \beta(y \cdot y'))$$

and $\beta\colon H^2(X,\mathbb{Z}/2\mathbb{Z}) \to H^3(X,\mathbb{Z})$ is the Bockstein homomorphism. The multiplicative structure of twisted K-theory alluded to above uses the fiberwise tensor product of bundles, respectively the tensor product over $C_0(X)$ of the corresponding section algebras and yields

$$K^n_{\mathcal{A}}(X) \otimes K^m_{\mathcal{A}'}(X) \to K^{n+m}_{\mathcal{A}\otimes\mathcal{A}'}(X) \ ,$$

which adds the classes of the twists.

The first two factors on the right hand side of (1.2) are related to the grading of the fibers. If we consider bundles with trivially graded fibers, i.e. the ungraded Brauer group $\mathrm{Br}(X)$, only the last factor survives. We have an isomorphism $\mathrm{Br}(X) \to \mathrm{Tor}(H^3(X,\mathbb{Z}))$ given by the Dixmier-Douady class of a bundle of matrix algebras $\mathcal{A} \to X$ constructed as follows: The automorphisms of $M_n(\mathbb{C})$ are all inner by a theorem of Skolem and Noether, i.e. $\mathrm{Aut}(M_n(\mathbb{C})) \cong PGL_n(\mathbb{C})$. Therefore this group fits into a central extension of the form

$$1 \to \mathbb{C}^* \to GL_n(\mathbb{C}) \to PGL_n(\mathbb{C}) \to 1 \ .$$

The Dixmier-Douady class is the obstruction against lifting the automorphism bundle of $\mathcal{A}$ to a principal $GL_n(\mathbb{C})$-bundle. It lies in the sheaf cohomology group $H^2(X,\underline{\mathbb{C}^*}) \cong H^3(X,\mathbb{Z})$ and is torsion by a theorem of Serre [67, Thm. 1.6]. This class vanishes if and only if $\mathcal{A}$ is the endomorphism bundle of a vector bundle, and it is additive with respect to the fiberwise tensor product, which explains why it factors over the equivalence relation (1.1) to a homomorphism $\mathrm{Br}(X) \to \mathrm{Tor}(H^3(X,\mathbb{Z}))$. Dixmier and Douady studied these classes in the more general context of continuous fields of elementary C^*-algebras [55], of which section algebras of matrix bundles are particular examples.

The section algebras are also contained in the class of continuous trace C^*-algebras. These are C^*-algebras that have Hausdorff spectrum, say X, and are locally Morita equivalent to $C_0(X)$. This observation was used by Rosenberg in [122] to extend the above definition to non-torsion twists classified by the full Brauer group $H^3(X,\mathbb{Z})$, not just the torsion part. Let $\mathbb{K}$ be the algebra of compact operators on a separable infinite dimensional Hilbert space H. The group of $*$-automorphisms $\mathrm{Aut}(\mathbb{K})$ is a topological group when equipped with the pointwise norm topology. It still just consists of inner automorphisms, i.e. it is isomorphic (as a topological group) to the projective unitary group $PU(H)$ equipped with the strong-$*$ operator topology. The unitary group $U(H)$ is contractible in this topology [54, Thm. 10.8.2], therefore $PU(H)$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z},2)$. Locally trivial algebra bundles with fiber $\mathbb{K}$ are thus classified by

$$[X, B\mathrm{Aut}(\mathbb{K})] \cong [X, BPU(H)] \cong [X, K(\mathbb{Z},3)] \cong H^3(X,\mathbb{Z}) \ .$$

The section algebras $C_0(X,\mathcal{A})$ of such bundles correspond precisely to the stable continuous trace C^*-algebras. If $\mathcal{A} \to X$ represents the class $[\mathcal{A}] \in H^3(X, \mathbb{Z})$, then the definition $K^n_{\mathcal{A}}(X) = K_n(C_0(X, \mathcal{A}))$ of the corresponding twisted K-groups is just the same as above.

Atiyah and Segal chose a slightly different approach to twisted K-theory in [8], which is closer to index theory and uses bundles of Fredholm operators. This point of view is motivated by a theorem of Atiyah and Jänich, which states that the space $\mathrm{Fred}(H)_{\|\cdot\|}$ of Fredholm operators on an infinite dimensional separable Hilbert space H equipped with the norm topology represents the group K^0, i.e. $K^0(X) \cong [X, \mathrm{Fred}(H)_{\|\cdot\|}]$. By Kuiper's theorem the unitary group equipped with the norm topology $U(H)_{\|\cdot\|}$ is contractible, hence if $PU(H)_{\|\cdot\|}$ denotes the projective unitary group equipped with the corresponding quotient topology, then we still have $BPU(H)_{\|\cdot\|} \simeq K(\mathbb{Z}, 3)$. The group $PU(H)_{\|\cdot\|}$ acts continuously on $\mathrm{Fred}(H)_{\|\cdot\|}$. Thus, if $\mathcal{A} \to X$ denotes the bundle of compact operators associated to a principal $PU(H)_{\|\cdot\|}$-bundle $P \to X$ and $\mathcal{F}_{\|\cdot\|} \to X$ is the associated bundle with fiber $\mathrm{Fred}(H)_{\|\cdot\|}$, we obtain an isomorphism $K^n_{\mathcal{A}}(X) \cong [X, \mathcal{F}_{\|\cdot\|}]$, where the right hand side denotes the homotopy classes of sections of $\mathcal{F}_{\|\cdot\|}$. The major disadvantage of this setup is, however, that it does not generalize easily to the case of equivariant twisted K-theory, since projective unitary actions of compact groups are often only strongly continuous and not norm-continuous.

Atiyah and Segal found an elegant solution for this problem: Instead of $\mathrm{Fred}(H)_{\|\cdot\|}$ they considered the space

$$\mathrm{Fred}'(H) = \{(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H) \mid AB - 1 \in \mathbb{K},\ BA - 1 \in \mathbb{K}\}$$

equipped with the subspace topology inherited from the embedding $(A, B) \mapsto (A, B, AB - 1, BA - 1)$ into $\mathcal{B}(H) \times \mathcal{B}(H) \times \mathbb{K} \times \mathbb{K}$, where $\mathcal{B}(H)$ denotes the bounded linear operators on H equipped with the compact-open topology and $\mathbb{K}$ are the compact operators with the norm topology. The topological group $PU(H)$ equipped as above with the strong-$*$ topology acts continuously on $\mathrm{Fred}'(H)$. Thus, if P is a principal $PU(H)$-bundle with associated bundle of compact operators $\mathcal{A}$ and associated $\mathrm{Fred}'(H)$-bundle $\mathrm{Fred}'(P)$, we have an isomorphism $K^*_{\mathcal{A}}(X) \cong [X, \mathrm{Fred}'(P)]$ by [8, Thm. 3.4]. Let G be a compact group acting continuously on the space X and choose $H = L^2(G) \otimes \ell^2(\mathbb{N})$. Let $P \to X$ be a G-equivariant principal $PU(H)$-bundle over X with local trivialisations that are compatible with the stabiliser groups [8, Section 6]. The equivariant twisted K-theory can now be defined as the homotopy classes of equivariant sections of the associated bundle $\mathcal{F} \to X$ with fiber $\mathrm{Fred}'(H)$. This definition was carefully studied and refined for discrete and proper actions in [10].

1.2 Applications of twisted K-theory

1.2.1 Loop groups and twisted K-theory

Let G be a compact, simple, connected and simply connected Lie group. In a series of three papers Freed, Hopkins and Teleman proved a surprising isomorphism between the equivariant twisted K-theory of G acting on itself by conjugation and the representation theory of the free loop group LG [62, 64, 63]. The assumptions on G imply that $H^3_G(G,\mathbb{Z}) \cong \mathbb{Z}$. An explicit equivariant twist $\mathcal{A} \to G$ representing the generator is constructed in [62, Sec. 2.4] in terms of a central $U(1)$-extension of a groupoid, but we will not go into the details here. Let $\mathcal{A}(k) = \mathcal{A}^{\otimes k}$, where we understand negative values of k to mean the $|k|$th tensor power of the bundle of opposite algebras. Let $h(G) \in \mathbb{Z}$ be the dual Coxeter number of G. By a positive energy representation of LG at level k we mean a unitary representation of $\widetilde{LG}_k \rtimes \mathbb{T}$, on a Hilbert space H, where $\widetilde{LG}_k$ denotes a central $U(1)$-extension of LG (these are parametrized by $k \in \mathbb{Z}$) and the torus group $\mathbb{T}$ acts via rotations. Furthermore, H satisfies the positive energy condition, which means that the weight decomposition

$$H = \bigoplus_{n\in\mathbb{Z}} H(n)$$

with $\theta \in \mathbb{T}$ acting by $e^{in\theta}$ on $H(n)$ satisfies $H(n) = 0$ for $n < 0$ and $\dim(H(n)) < \infty$ for $n \geq 0$. Denote by $\mathrm{Rep}^{\mathrm{pos}}_k(LG)$ the additive group completion of the monoid of isomorphism classes of positive energy representations at level k with respect to the direct sum. This group is in fact a ring with respect to the fusion product which preserves the level. It is called the Verlinde ring. The main theorem of [64] states the following:

$$K^{\dim(G)}_{G,\mathcal{A}(k+h(G))}(G) \cong \mathrm{Rep}^{\mathrm{pos}}_k(LG)\ .$$

This is an isomorphism of rings, where the multiplicative structure on the left hand side arises from the push-forward with respect to the group multiplication $\mu\colon G \times G \to G$.

The above theorem is particularly interesting in light of Chern-Simons theory, since it proves that its dimension reduction is related to twisted K-theory. To see what this means, let $\mathcal{C}ob_{n,n-1}$ be the category, which has $(n-1)$-dimensional closed smooth manifolds as its objects and n-dimensional cobordisms between them as its morphisms. The composition is given by gluing[1]. The category $\mathcal{C}ob_{n,n-1}$ is a monoidal category, where the tensor product is defined to be the disjoint union and the unit object is the empty set. Classically, a topological quantum field theory is a monoidal functor $Z\colon \mathcal{C}ob_{n,n-1} \to \mathcal{V}ect_K$, where $\mathcal{V}ect_K$ denotes the

[1] The details of the precise construction of $\mathcal{C}ob_{n,n-1}$ are quite technical. Since we just aim at giving a sketch of the setup here, we omit them.

linear category of K-vector spaces. The category $\mathcal{C}ob_{n,n-1}$ can be extended to a 2-category $\mathcal{C}ob_{n,n-1,n-2}$, which has closed smooth $(n-2)$-manifolds as its objects, $(n-1)$-cobordisms as 1-morphisms and n-cobordisms between the $(n-1)$-cobordisms as 2-morphisms (these have to be manifolds with corners). Likewise, there is a natural extension of the right hand side given by the 2-category $\mathcal{C}at_K$ of K-linear categories. Again, both sides can be equipped with suitable monoidal structures. An extended (3-tier) topological field theory consists of a monoidal functor $Z\colon \mathcal{C}ob_{n,n-1,n-2} \to \mathcal{C}at_k$. In the case $n = 3$ Reshetikhin and Turaev proved that $Z(S^1)$ is a modular tensor category, which completely determines the functor Z. They also managed to construct a 3-tier TQFT starting from a given modular tensor category [143, Chap. IV]. Since the category of positive energy representations of a Lie group G as above is modular, there is a corresponding TQFT with $K = \mathbb{C}$ associated to it, which is Chern-Simons theory. Given a 3-tier TQFT Z it is easy to construct an ordinary TQFT from it via $Z'(M) = Z(M \times S^1)$. This process is called dimension reduction. For $n = 3$ this will be a 2-dimensional TQFT. These have been fully classified [1]. Again they are completely determined by the vector space $Z'(S^1)$, which turns out to be a Frobenius algebra. In the case of Chern-Simons Theory the dimension reduction is the complex Verlinde ring and the Freed-Hopkins-Teleman theorem can be seen as the statement that twisted K-theory provides its integral refinement [61].

1.2.2 D-Branes and twisted K-theory

Another application of twisted K-theory is also closely related to theoretical physics. To motivate the idea, we consider classical electromagnetism first: The (vacuum) Maxwell equations on a spacetime M with Lorentzian metric g can be written as $dF = 0$ and $d^*F = 0$ for a 2-form $F \in \Omega^2(M)$. They are obtained by minimizing the action

$$S = \int_M F \wedge *F \ .$$

The 2-form F is called the strength of the electromagnetic field. However, it does not describe the field completely as it does not account for topological effects like the Aharanov-Bohm effect. The more fundamental object is a hermitian line bundle $L \to M$ together with a unitary connection that induces a parallel transport

$$A_\gamma\colon L_x \to L_y$$

for a smooth path $\gamma\colon I \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. The field strength corresponds to the curvature of the connection on L and can be obtained from A by applying it to infinitesimally small closed curves.

A similar situation occurs in string theory [130]: By general relativity the pseudo-riemannian metric g of a spacetime M satisfies Einstein's field equations, which can be reformulated by saying that it is a critical point of the action

$$S(M,g) = \int_M R \, d\mathrm{vol}_g$$

where R is the scalar curvature of g. The low energy expansion of the action in string theory contains the above term as its zeroth order and can be written as

$$S_{\mathrm{low}}(M,g,H) = \int_M (R \, d\mathrm{vol}_g + H \wedge *H)$$

with $H \in \Omega^3(M)$. Following the above philosophy, H should again be a curvature of a connection on some bundle. However, since it is a 3-form it does not fit into the classical setup. This lead to the definition of gerbes, which are generalizations of line bundles. Isomorphism classes of them are classified by $H^3(M,\mathbb{Z})$ instead of $H^2(M,\mathbb{Z})$. The gerbe that appears in string theory is called the B-field on M and can also be viewed as a line bundle over the loop space LM. It has $H \in \Omega^3(M)$ as its 3-curvature, which turns out to be closed and represents the image of the class of the B-field in $H^3(M,\mathbb{R})$. Note that in this picture we can also view the B-field as a twist of K-theory over M.

The connection on the B-field yields a notion of parallel transport along surfaces Σ in M (called world sheets in string theory). In this picture Σ should be seen as a generalized path in M from a set of ingoing boundary circles to outgoing boundary circles. Instead of a moving point, Σ describes moving closed strings in M. In recent years string theorists also considered moving open strings (i.e. intervals) in M. In this case, the submanifolds $Y \subset M$, on which strings are allowed to start respectively end are called D-Branes[2].

If the B-field is trivial, then the D-Branes come equipped with vector bundles carrying connections. They describe the boundary conditions at the end of the string and their class in K-theory is called the charge of the D-Brane [152]. This is no longer true in the presence of a non-trivial B-field, where the transition data for the vector bundle over the D-Brane is modified by the gerbe. However, this data still yields a class in twisted K-theory, where the twist is classified by the Dixmier-Douady invariant of the gerbe in $H^3(M,\mathbb{Z})$.

These observations motivated the definition of bundle gerbes [21]. They can be described as central $U(1)$-extensions of certain groupoids. Modules over bundle gerbes yield an elegant geometric picture of twisted K-theory in case of torsion twists. The classes here are represented by twisted versions of vector bundles as described above.

[2]The D stands for Dirichlet as in "Dirichlet boundary conditions".

1.2.3 Topological T-Duality

As we have seen in the last section, twisted K-theory arises quite naturally when dealing with the low energy approximation of string theory. Given a spacetime M with pseudo-riemannian metric g and B-field B, string theory associates – at least heuristically – a conformal field theory to the triple (M, g, B). It is interesting to note that different spacetimes may in fact produce the same field theory this way. This happens for example for the flat torus $\mathbb{T}^n$ with trivial B-field and the dual torus $\mathbb{T}^{n*}$, i.e. the quotient of the dual euclidean space by the dual lattice (this procedure changes for example the radius of a circle from r to $1/r$).

If we consider the above situation as the local instance of a global phenomenon, we might wonder what will happen if M is not the torus itself, but a bundle with fiber $\mathbb{T}^n$ to which we apply the above duality fiberwise. Since many spacetimes in string theory have torus fibers in form of compactified dimensions, this is more than just a toy example. If we forget about the metric and just consider the topological data, which consists of pairs of a torus bundle together with a B-field, we obtain topological T-Duality [27]. Even though it was motivated by string theory, it has developed a life of its own outside the realm of theoretical physics with important applications in mathematics [52, 132].

As we will see, topological T-Duality fits quite naturally into the non-commutative framework of twisted K-theory: Fix a compact Hausdorff space X and let A be a stable continuous trace C^*-algebra, such that its spectrum P is a principal $\mathbb{T}^n$-bundle over X. Since A is isomorphic to the section algebra of a bundle $\mathcal{K} \to P$ with fiber $\mathbb{K}$, this is a valid model for a space P with a twist of K-theory over P. We call two such algebras A and A' T-Dual to each other if the induced $\mathbb{R}^n$-action on P (with stabilizer $\mathbb{Z}^n$) lifts to an action of $\mathbb{R}^n$ on A and we have $A' \cong A \rtimes \mathbb{R}^n$.

The question of whether A has a T-dual therefore boils down to a lifting problem for the $\mathbb{R}^n$-action on A and a classification problem for the the crossed product. In fact, if the action lifts and the crossed product is a stable continuous trace C^*-algebra, it follows that the spectrum of $A' = A \rtimes \mathbb{R}^n$ is again a principal $\mathbb{T}^n$-bundle $P' \to X$ with a canonical $\mathbb{R}^n$-action on A' such that $A' \rtimes \mathbb{R}^n \cong A$ [97, Thm. 3.1].

1.3 Higher twisted K-theory and C^*-algebras

If we ignore the graded twists for simplicity and restrict to the ones classified by the group $H^3(X, \mathbb{Z})$, then we can summarize the properties of twisted K-theory introduced in the first section as follows:

(a) Twisted K-theory $X \mapsto K^*_{\mathcal{A}}(X)$ is a homotopy invariant functor from the category of bundles of compact operators over locally compact topological spaces to graded abelian groups. There is a corresponding version of this functor for pairs and it has long exact sequences, Mayer-Vietoris sequences, Bott periodicity and satisfies excision.

(b) The twisted K-groups $K^*_{\mathcal{A}}(X)$ are modules over untwisted K-theory $K^*(X)$ in a natural way.

(c) The isomorphism classes of twists form a group, the full Brauer group $H^3(X, \mathbb{Z})$, with respect to the fiberwise tensor product.

From the point of view of homotopy theory the twists classified by the full Brauer group $H^3(X, \mathbb{Z})$ are not the most general ones. This was already pointed out by Atiyah and Segal in [8].

To get an intuition for what the most general twists look like, observe that $PU(H) \simeq K(\mathbb{Z}, 2)$ is a model for the classifying space of hermitian line bundles. In fact, the line bundle obtained from a map $f \colon X \to PU(H)$ is associated to the pullback of the universal principal $U(1)$-bundle $U(H) \to PU(H)$. Now let $\mathcal{A} \to X$ be a bundle of compact operators and denote by $P \to X$ its principal $PU(H)$-bundle. Choosing a trivialising cover $(U_i)_{i \in I}$ of X and local trivialisations for P over the U_i, we obtain transition maps $U_{ij} \to PU(H)$ (with $U_{ij} = U_i \cap U_j$) giving line bundles $L_{ij} \to U_{ij}$. Over triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ there is a multiplication map $\mu_{ijk} \colon L_{ij} \otimes L_{jk} \to L_{ik}$, and there is an associativity constraint over U_{ijkl}. This is precisely the data of a gerbe.

A line bundle $L \to X$ represents an invertible element in $K^0(X)$ and the fact that we can twist K-theory by $PU(H)$-bundles is a "delooped version" of the observation that forming the tensor product with a line bundle yields a $K^*(X)$-module isomorphism $K^*(X) \to K^*(X)$. This also makes sense from another point of view: If R is a commutative ring, the classifying space for locally trivial bundles of R-modules with fibers isomorphic to R is $BGL_1(R)$, where $GL_1(R)$ is the group of units of the ring. With this in mind, twisting K-theory by more general invertible elements in $K^0(X)$ seems to be the natural generalization. It is an exercise to check that these invertibles are precisely the ones that are represented by *virtual line bundles*, i.e. classes $[E] - [F] \in K^0(X)$, such that $\dim(E) - \dim(F) \in \{\pm 1\}$.

Let BU be the colimit over $\mathbb{N}$ of the classifying spaces $BU(n)$. This is the classifying space of stable hermitian vector bundles. There is a natural isomorphism $K^0(X) \cong [X, KU]$, where $KU = BU \times \mathbb{Z}$ and the $\mathbb{Z}$-factor takes care of the virtual dimension. Let $GL_1(KU) = BU \times \{\pm 1\}$. If $GL_1(K^0(X))$ denotes the units in the ring $K^0(X)$, we have

$$GL_1(K^0(X)) = [X, GL_1(KU)] \ .$$

Notice that the group operation on both sides is induced by the tensor product of virtual line bundles, which turns $GL_1(KU)$ into an H-space. The classifying space for the twists of K-theory should be $BGL_1(KU)$, but to make sense of this, we need to ensure that the H-space structure is associative enough. It needs to be an A_∞- or even better an E_∞-space.

It was proven by Segal in [129] that the tensor product H-space structure on BU can in fact be extended to that of an E_∞-space. It is straightforward to modify his argument to work for the whole space $GL_1(KU)$ as well. Thus, $GL_1(KU)$ is an infinite loop space, which implies that we obtain a unit spectrum of K-theory via succesive deloopings of $GL_1(KU)$. It was shown later by May, Quinn and Ray that any E_∞-ring spectrum has an analogous unit spectrum [104] and for highly structured ring spectra such as symmetric ones, Sagave and Schlichtkrull found models for these unit spectra in terms of commutative $\mathcal{I}$-monoids [124, 123]. Another approach to unit spectra and twists of K-theory and TMF was developed by Ando, Blumberg and Gepner in [5] and by Ando, Blumberg, Gepner, Hopkins and Rezk in [6]. This uses the language of stable ∞-categories.

Thus, the group $[X, BGL_1(KU)]$ generalizes the (graded) full Brauer group of X and in this homotopy theoretic picture a (higher) twist corresponds to a map $f\colon X \to BGL_1(KU)$. The corresponding twisted K-groups $K^*_f(X)$ can be defined using the parametrized stable homotopy theory as developed by May and Sigurdsson [102] or via generalized Thom spectra as in [5, Sec. 5]. We will not go into the details here.

What is missing in both cases is a description of the higher twists via non-commutative geometry, i.e. in terms of bundles of C^*-algebras, which was one of the main motivations for this thesis. To make this precise, we can formulate the following questions as a guideline:

Questions. *(a) Just as $B\mathrm{Aut}(\mathbb{K}) \simeq BPU(H) \simeq K(\mathbb{Z}, 3)$ is a model for the twists classified by $H^3(X, \mathbb{Z})$, are there C^*-algebras A, such that $B\mathrm{Aut}(A \otimes \mathbb{K}) \simeq BGL_1(KU)$?*

(b) The torsion elements in $H^3(X, \mathbb{Z})$ are represented by bundles of matrix algebras instead of compact operators. If A is an algebra as in (a), is it still true that torsion elements in $[X, B\mathrm{Aut}(A \otimes \mathbb{K})]$ can be represented by bundles with fiber $M_n(A)$?

(c) If $\mathcal{A} \to X$ is a locally trivial bundle of C^*-algebras with fiber $A \otimes \mathbb{K}$ as in (a), we can define higher twisted K-theory via $K^*_{\mathcal{A}}(X) = K_*(C(X, \mathcal{A}))$. How does this definition relate to the homotopy theoretic ones?

1.4 Outline of the papers

The papers outlined below take the above questions as a starting point and give complete answers to all of them in terms of strongly self-absorbing C^*-algebras. In this section, their main theorems will be indicated and put into context.

1.4.1 A Dixmier-Douady theory for strongly self-absorbing C^*-algebras

To address question (a) we have to find a C^*-algebra A such that $B\mathrm{Aut}(A \otimes \mathbb{K})$ is an infinite loop space with respect to the fiberwise tensor product of locally trivial bundles. For this to make sense it is necessary that A is self-absorbing, i.e. $A \otimes A \cong A$. This generalizes the situation for $A = \mathbb{C}$, in which case we get $B\mathrm{Aut}(\mathbb{K}) = BPU(H) \simeq K(\mathbb{Z}, 3)$. Just as in this case, we also want that the trivial bundle provides a neutral element for the monoid $\mathcal{B}un_X(A \otimes \mathbb{K})$ of isomorphism classes of locally trivial bundles with fiber $A \otimes \mathbb{K}$ over X. These considerations lead quite naturally to the following class of C^*-algebras, which was first studied by Toms and Winter [139] and plays a fundamental role in Elliott's classification program.

Definition 1.4.1. A separable unital C^*-algebra A is called *strongly self-absorbing* if there is an isomorphism $\psi \colon A \to A \otimes A$ and a continuous path $u \colon [0, 1) \to U(A \otimes A)$ into the unitary group of $A \otimes A$, such that $\lim_{t \to 1} \|\psi(a) - u_t(a \otimes 1_A)u_t^*\| = 0$.

The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are strongly self-absorbing, as are the Jiang-Su algebra $\mathcal{Z}$ and all infinite UHF-algebras. The class is closed under taking tensor products.

Let A be a strongly self-absorbing C^*-algebra. The main theorems of chapter 2 reveal that $\mathcal{B}un_X(A \otimes \mathbb{K})$ is in fact an abelian group coming from a generalized cohomology theory just as the Brauer group $\mathrm{Br}(X) \cong H^3(X, \mathbb{Z})$ arises from ordinary cohomology. Of course, we obtain $\mathrm{Br}(X)$ in the special case $A = \mathbb{C}$. More precisely we have:

Theorem 1.4.2. *Let X be a compact metrizable space and let A be a strongly self-absorbing C^*-algebra.*

(a) $B\mathrm{Aut}(A \otimes \mathbb{K})$ *is an infinite loop space with respect to the tensor product (Cor. 2.3.7).*

(b) The monoid $\mathcal{B}un_X(A \otimes \mathbb{K})$ of isomorphism classes of locally trivial bundles over X with fiber $A \otimes \mathbb{K}$ is an abelian group. $B\mathrm{Aut}(A \otimes \mathbb{K})$ *is the first space in a spectrum defining a cohomology theory $E_A^\bullet$ with $E_A^1(X) = \mathcal{B}un_X(A \otimes \mathbb{K})$ (Thm. 2.3.8).*

Further evidence that $\mathcal{B}un_X(A \otimes \mathbb{K})$ is the correct generalization of the Brauer group is provided by the group homomorphism $\mathrm{Aut}(\mathbb{K}) \to \mathrm{Aut}(A \otimes \mathbb{K})$ given by $\alpha \mapsto \mathrm{id}_A \otimes \alpha$. On classifying spaces it induces a map $B\mathrm{Aut}(\mathbb{K}) \to B\mathrm{Aut}(A \otimes \mathbb{K})$ giving a natural transformation $H^3(X, \mathbb{Z}) \to \mathcal{B}un_X(A \otimes \mathbb{K})$. We also prove that $\mathrm{Aut}(A)$ is contractible in the pointwise norm topology (Thm. 2.2.3), which yields a generalization of [51, Cor. 3.1].

Instead of section algebras of locally trivial bundles Dixmier and Douady studied continuous fields of elementary C^*-algebras in [55]. Continuous fields (also called continuous $C_0(X)$-algebras) were introduced by Kasparov in [84]. As the name suggests, they are C^*-algebras A carrying an action of $C_0(X)$ for a locally compact Hausdorff space X such that for all $a \in A$ the norm function $x \mapsto \|a(x)\|$ is continuous. Section algebras are examples, but the notion is much more general and closer in spirit to sheaf theory.

It was shown in [55] that a continuous $C_0(X)$-algebra B with fibers $\mathbb{K}$ is isomorphic to the sections of a locally trivial bundle if and only if B satisfies Fell's condition. We were able to generalize this fact to our setup in the following way: Let A be strongly self-absorbing. We say that a continuous $C_0(X)$-algebra B with fiber $A \otimes \mathbb{K}$ satisfies the generalized Fell condition if for every point $x \in X$ there exists a closed neighborhood V and a projection $p_V \in B(V)$, such that $p_V(x) \in GL_1(K_0(B(x)))$ for all $x \in V$ (Def. 2.4.1). For $A = \mathbb{C}$ we obtain the original Fell condition, which ensures the existence of local rank 1-projections for every point.

In the second main result of that paper, we could prove that for locally compact spaces X of finite covering dimension the generalized Fell condition is equivalent to local triviality for continuous $C_0(X)$-algebras with stabilized strongly self-absorbing fibers (Thm. 2.4.2).

The generalized cohomology theory $E^*_A(X)$ is computable via the Atiyah-Hirzebruch spectral sequence. These computations enable us to define rational characteristic classes in case A satisfies the universal coefficient theorem (which is conjectured to be true for all strongly self-absorbing C^*-algebras). Let $M_{\mathbb{Q}}$ be the universal UHF-algebra with $K_0(M_{\mathbb{Q}}) \cong \mathbb{Q}$ and $K_1(M_{\mathbb{Q}}) = 0$. The construction of the class $\delta_k \in H^{2k+1}(X, \mathbb{Q})$ for $k \geq 1$ is based on the observation that $A \otimes M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \otimes \mathbb{K}$, which induces a map

$$\mathcal{B}un_X(A \otimes \mathbb{K}) \to \mathcal{B}un_X(M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \otimes \mathbb{K}) \cong H^1(X, \mathbb{Q}^\times) \oplus \prod_{n \in \mathbb{N}} H^{2k+1}(X, \mathbb{Q})$$

and δ_k is given by projecting to the corresponding factor of the product. The existence of these characteristic classes and the E^2-term of the Atiyah-Hirzebruch spectral sequence support the conjecture that $B\mathrm{Aut}(A \otimes \mathbb{K})$ is the first space of the unit spectrum of the multiplicative cohomology theory $X \mapsto K_*(C_0(X) \otimes A)$, which will be proven in the next paper.

publication status: This is a joint paper with Marius Dadarlat that was accepted for publication by Journal für die reine und angewandte Mathematik [47].

1.4.2 Unit spectra of K-theory from strongly self-absorbing C^*-algebras

Let A be a strongly self-absorbing C^*-algebra. Consider the contravariant functor $X \mapsto K^*_A(X) := K_*(C_0(X) \otimes A)$ from locally compact Hausdorff spaces to graded abelian groups. This is a generalized cohomology theory, which has a multiplicative structure induced by

$$\mu \colon K_*(C_0(X) \otimes A) \otimes K_*(C_0(X) \otimes A) \to K_*(C_0(X \times X) \otimes A \otimes A) \to K_*(C_0(X) \otimes A)$$

where the first map is given by the tensor product on K-theory and the second uses the pullback via the diagonal map $\Delta^*\colon C_0(X \times X) \to C_0(X)$ together with an isomorphism $A \otimes A \to A$. The homomorphism μ does not depend on the choice of this isomorphism by the contractibility of $\mathrm{Aut}(A)$ (Thm. 2.2.3). In the cases $A \in \{\mathbb{C}, \mathcal{Z}, \mathcal{O}_\infty\}$ the above functor coincides with topological K-theory. If A is an infinite UHF-algebra, it agrees with a localization of K-theory and for $A = M_{\mathbb{Q}}$ we obtain its rationalization. For $A = \mathcal{O}_2$ the functor associates the trivial group to every space.

To give a complete answer to question (a) we define a commutative symmetric ring spectrum KU^A representing $X \mapsto K_A^*(X)$, find a suitable model for its unit spectrum and compare $B\mathrm{Aut}(A \otimes \mathbb{K})$ and $BGL_1(KU^A)$ via a zig-zag of continuous maps.

For the definition of KU^A we can build upon previous work by Joachim [79] and Higson and Guentner [69]. They constructed a commutative symmetric ring spectrum representing K-theory. A slight modification yields the desired model KU^A for $K_A^*(X)$ (Def. 3.4.1). We use commutative $\mathcal{I}$-monoids to describe the unit spectrum of KU^A as in [124]. These are functors from a small model for the category of finite sets and injective maps to topological spaces together with a multiplication map and a unit (Def. 3.2.2). Each commutative $\mathcal{I}$-monoid M has a Γ-space $\Gamma(M)$ in the sense of Segal associated to it [129].

We adopt the notation from [124] and denote the $\mathcal{I}$-monoid model for the unit spectrum by $\Omega^\infty(KU^A)^*$. Likewise, there is an $\mathcal{I}$-monoid for the automorphism group, which we call $G_A\colon \mathcal{I} \to \mathcal{G}rp$ and which is defined by $G_A(\mathbf{n}) = \mathrm{Aut}((A \otimes \mathbb{K})^{\otimes n})$. The natural action of $\mathrm{Aut}(A \otimes \mathbb{K})$ on KU^A induces a map of $\mathcal{I}$-monoids $G_A \to \Omega^\infty(KU^A)^*$. For strongly self-absorbing $A \neq \mathbb{C}$ it yields an isomorphism on all π_n for $n > 0$ and is the inclusion of positive invertible elements into all invertible elements on π_0 (Thm. 3.4.6). In particular, it is a weak equivalence if A is in addition purely infinite.

The only problem with this comparison is that there are two H-space structures on $\mathrm{Aut}(A \otimes \mathbb{K})$: The first one is just composition, the second one uses the tensor product of automorphisms and conjugation by an isomorphism $(A \otimes \mathbb{K})^{\otimes 2} \cong A \otimes \mathbb{K}$. If we denote the first one by ν and the second by μ, then we have two classifying spaces $B_\nu G_A$ and $B_\mu G_A$. The first one is a model for $B\mathrm{Aut}(A \otimes \mathbb{K})$, i.e. it classifies locally trivial bundles with fiber $A \otimes \mathbb{K}$ while the second one is (almost) weakly equivalent to $BGL_1(KU^A)$ via the map $G_A \to \Omega^\infty(KU^A)^*$ as discussed above (and in Thm. 3.3.8). To show that both agree as infinite loop spaces, we note that

$$(\alpha \otimes \beta) \circ (\alpha' \otimes \beta') = (\alpha \circ \alpha') \otimes (\beta \circ \beta')$$

for $\alpha, \alpha', \beta, \beta' \in \mathrm{Aut}(A \otimes \mathbb{K})$. This is a graded version of the Eckmann-Hilton condition. We therefore define Eckmann-Hilton $\mathcal{I}$-groups (Def. 3.3.1), keeping G_A in mind as our primary example. The main theorem about EH-$\mathcal{I}$-groups is Thm. 3.3.6 stating that the above condition suffices to ensure that $B_\nu G_A$ and $B_\mu G_A$ agree stably up to homotopy. Since the infinite loop space structure on $B\mathrm{Aut}(A \otimes \mathbb{K})$ was defined using a permutative category in chapter 2, we have to compare these two approaches as well, which is done in Thm. 3.3.13.

Finally, we are in the position to answer question (a) and identify the infinite loop spaces for all known strongly self-absorbing C^*-algebras. These results are summarized in the main applications – Thm. 3.4.7 and Thm. 3.4.8. In short, we have $BGL_1(KU) \simeq B\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})$, $BBU_\otimes \simeq B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})$ and $BGL_1(KU[1/p]) \simeq B\mathrm{Aut}(M_{p^\infty} \otimes \mathcal{O}_\infty \otimes \mathbb{K})$.

publication status: This is a joint paper with Marius Dadarlat that appeared in print in Algebraic & Geometric Topology [49].

1.4.3 The Brauer group in generalized Dixmier-Douady theory

Let D be a strongly self-absorbing C^*-algebra[3]. In general, the group $\mathrm{Aut}(D \otimes \mathbb{K})$ is not connected. Let $\mathrm{Aut}_0(D \otimes \mathbb{K})$ be the connected component of the identity. Locally trivial bundles with structure group $\mathrm{Aut}_0(D \otimes \mathbb{K})$ correspond to a natural subgroup $\bar{E}^1_D(X) \subset E^1_D(X) \cong \mathcal{B}un_X(D \otimes \mathbb{K})$, which is also denoted by $\mathscr{C}^0_{D\otimes\mathbb{K}}(X)$ in the paper. It is obtained from a generalized cohomology theory, namely the first connected cover of $g\ell_1(KU^D)$. We call such bundles orientable. A bundle $\mathcal{A} \to X$ in $\mathcal{B}un_X(D \otimes \mathbb{K})$ is orientable if and only if the associated bundles of K_0-groups, obtained by taking the K-theory functor fiberwise, is trivialisable. To give some examples, we have

$$\bar{E}^1_{\mathcal{O}_\infty}(X) \cong \bar{E}^1_{\mathcal{Z}}(X) \cong [X, BBU_\otimes] \ .$$

The classical (ungraded) Brauer group consists of the torsion elements in $\bar{E}^1_{\mathbb{C}}(X) = H^3(X, \mathbb{Z})$, which are represented by bundles of matrix algebras. In chapter 4 we answer question (b) concerning whether the same holds true in generalized Dixmier-Douady theory: Are torsion elements in $\bar{E}^1_D(X)$ represented by locally trivial continuous fields with fiber $M_n(D)$ for some $n \in \mathbb{N}$? The main result of that chapter is Thm. 4.2.10, which states that for an orientable locally trivial continuous field A the following statements are equivalent:

(i) A represents a torsion element in $\bar{E}^1_D(X)$,

(ii) all rational characteristic classes $\delta_k(A)$ vanish,

(iii) there exists a locally trivial continuous field B over X with fibers isomorphic to $M_n(D)$ for some $n \in \mathbb{N}$ such that $B \otimes \mathbb{K} \cong A$.

[3] In this chapter we changed the notation from A to D, which is more common in the literature.

The order of the torsion element and the dimension n of the matrix algebra are related as well. More precisely, if the order is n^k for some $k \in \mathbb{N}$, then B can be chosen to have fiber isomorphic to $M_{n^l}(D)$ for some $l \in \mathbb{N}$ as is proven in Thm. 4.2.11.

The generalized Brauer group $\mathrm{Br}_D(X)$ consisting of all torsion elements in $\bar{E}^1_D(X)$ can be defined using an equivalence relation similar to (1.1): Two locally trivial continuous fields A_i, $i \in \{1,2\}$ are equivalent, if they become isomorphic after forming tensor products with the endomorphism algebras of locally trivial fields of finitely generated projective right Hilbert D-modules (Def. 4.2.13). In the case $A = \mathbb{C}$ this recovers (1.1).

However, not every construction from the classical theory carries over to the generalized setting so easily: It is straightforward to see that the inverse of a class $[\mathcal{A}] \in H^3(X,\mathbb{Z})$ represented by $\mathcal{A} \to X$ with fibers $\mathbb{K}$ is given by the bundle $\mathcal{A}^{\mathrm{op}} \to X$ of opposite algebras. We calculate the effect of this operation on the rational characteristic classes in Thm. 4.3.4 and give an example of a locally trivial continuous field, where the inverse in the $\bar{E}^1_D(X)$ is not given by the opposite bundle in Ex. 4.3.5.

publication status: This chapter consists of a joint paper with Marius Dadarlat that appeared in the Journal of non-commutative geometry [48].

1.4.4 A noncommutative model for higher twisted K-theory

Let D be a strongly self-absorbing and purely infinite C^*-algebra. The outcome of the first two chapters is a homotopy equivalence $BGL_1(KU^D) \simeq B\mathrm{Aut}(D \otimes \mathbb{K})$, which identifies $E^1_D(X) \cong [X, B\mathrm{Aut}(D \otimes \mathbb{K})]$ with the group of twists of the multiplicative cohomology theory $X \mapsto K^*_D(X)$. If D is strongly self-absorbing, but not purely infinite, we obtain a subgroup of $[X, BGL_1(KU^D)]$. Just as in the case of bundles of compact operators, we now expect the higher twisted K-groups to be given by

$$K^*_{\mathcal{A}}(X) = K_*(C_0(X,\mathcal{A})) \tag{1.3}$$

where $\mathcal{A} \to X$ is a locally trivial bundle of C^*-algebras with fiber $D \otimes \mathbb{K}$. The goal of chapter 5 is to compare this definition of $K^*_{\mathcal{A}}$ with the homotopy theoretic ones in the literature, i.e. to answer question (c).

To obtain a symmetric spectrum representing $K^*_{\mathcal{A}}$ we first define a module spectrum $KU^{D,\mathrm{mod}}$ over KU^D, which is equivalent to KU^D as a module over itself and carries an action of $\mathrm{Aut}(D \otimes \mathbb{K})$ (see Def. 5.2.3). We then define a bundle with fiber $KU^{D,\mathrm{mod}}$ over $B\mathrm{Aut}(D \otimes \mathbb{K})$ by $\mathscr{K}U^D_n = E\mathrm{Aut}(D \otimes \mathbb{K}) \times_{\mathrm{Aut}(D \otimes \mathbb{K})} KU^{D,\mathrm{mod}}_n$. Let $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ be a continuous map classifying a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle $\mathcal{P} \to X$. The higher twisted

K-groups[4] $K^*_{\mathcal{P}}(X,B)$ for a pair of compact Hausdorff spaces are defined as the (stable) homotopy groups of the corresponding space of sections of $f^*\mathscr{K}U_n^D$ (see Def. 5.2.6).

In Thm. 5.2.7 we prove that this functor has all the properties to be expected from twisted K-theory: homotopy invariance, K^*_D-module structure, six-term exact sequences and excision. Moreover, we have a natural isomorphism between $K^*_{\mathcal{P}}(X,\emptyset)$ and (1.3) (where $\mathcal{A}$ is understood to be the algebra bundle associated to $\mathcal{P}$).

The definition of twisted K-theory via parametrized stable homotopy theory [102] uses orthogonal spectra. Instead of developing the analogue of the qf-model structure for symmetric spectra, which takes a considerable amount of work in [102] , we restrict the comparison to the cases $D \in \{\mathbb{C}, \mathcal{O}_\infty, \mathcal{Z}\}$. In these cases we have good orthogonal models. In principle these models would also work for general D at the expense of replacing the KU^D-module structure with just the one over $KU^{\mathbb{C}}$. It is proven in Thm. 5.2.14 that the two versions of twisted K-theory agree.

Whereas the comparison of the different versions of twisted K-theory was quite straightforward, the arguments for the dual theory are more involved. Motivated by (1.3) we define analytic twisted K-homology to be the functor $X \mapsto K_*^{\mathcal{P}}(X) = KK_{-*}(C_0(X,\mathcal{A}), D)$, where $D \otimes \mathbb{K}$ is the fiber of $\mathcal{A}$. The topological version of twisted K-homology from [5] uses a generalized Thom spectrum. Since we have a bundle of symmetric spectra at hand, this boils down to Def. 5.3.1 for finite CW-complexes X.

The comparison of these two functors is based on a Mayer-Vietoris argument and the proof for the untwisted case given by Baum, Higson and Schick in [12]. The idea is to use framed bordism to define an intermediate generalized homology theory, called $k^{\mathcal{P}}$ (see Def. 5.3.11), which maps isomorphically to topological as well as analytic K-homology. A subtle detail in the comparison of $k^{\mathcal{P}}$ with its analytic counterpart is that the transformation uses a twisted version of Poincare duality proven by Echterhoff, Emerson and Kim [57]. This duality inverts the twists, which – as we have seen above – is more complicated than just taking the opposite bundle. We circumvent these problems by building inverses into the extra data, when defining our source category $\mathcal{CW}_D^{\mathrm{fin}}$ in Def. 5.3.8. The main result is Thm. 5.3.18, where the two functors are compared for strongly self-absorbing C^*-algebras D that satisfy the UCT. It is shown that they are naturally isomorphic.

We conclude the chapter by giving a detailed description of the higher twists for spaces that are suspensions. In this case the twists are parametrized by invertible elements in K-theory and can be obtained from linking algebras of Morita equivalences. We compute the twisted K-groups for these spaces, in particular for all spheres in Thm. 5.4.5 and Cor. 5.4.6.

[4]In this chapter we label the twisted K-groups by $\mathcal{P}$ instead of $\mathcal{A}$, which is just a notational issue.

publication status: The material in this chapter was accepted for publication by the Journal of Topology [111].

1.4.5 Deformations of nilpotent groups and homotopy symmetric C^*-algebras

The material in this chapter is not directly related to twisted K-theory. However, it fits into the theme of applications of homotopy theory to operator algebras. The definition of strongly self-absorbing C^*-algebras can be rephrased into the statement that there exists an isomorphism $D \to D \otimes D$, which is *asymptotically unitarily equivalent* to the embedding $d \mapsto d \otimes 1_D$. Even though this looks very similar to unitary equivalence it is a much more flexible notion. More generally, we can consider asymptotic morphisms, which are families of maps $\varphi_t \colon A \to B$ between two C^*-algebras A and B parametrized by the half-line $[1, \infty)$, such that they satisfy the axioms of a $*$-homomorphism in the limit $t \to \infty$, i.e.

(i) for all $a \in A$ the map $t \mapsto \varphi_t(a)$ is norm-continuous and bounded,

(ii) $\lim_{t\to\infty} \|\varphi_t(a + \lambda\, b) - (\varphi_t(a) + \lambda\, \varphi_t(b))\| = 0$, $\lim_{t\to\infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| = 0$,

(iii) $\lim_{t\to\infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0$.

Let $[[A, B]]$ denote the homotopy classes of asymptotic morphisms, $[[A, B]]^{\mathrm{cp}}$ are the homotopy classes of completely positive contractive asymptotic morphisms and $[[A, B]]_{\mathbb{N}}$ are homotopy classes of discrete asymptotic morphisms parametrized by $\mathbb{N}$ instead of $[1, \infty)$. The importance these notions is underlined by the fact that E-theory as well as KK-theory may be expressed by them. More precisely, we have [34, 73]:

$$E(A, B) \cong [[SA, SB \otimes \mathbb{K}]] \quad , \quad KK(A, B) \cong [[SA, SB \otimes \mathbb{K}]]^{\mathrm{cp}} .$$

The suspension functor provides a monoid homomorphism $S \colon [[A, B \otimes \mathbb{K}]] \to E(A, B)$ and similarly for KK-theory. The question, which E-theory classes may be naturally represented by asymptotic morphisms $A \to B \otimes \mathbb{K}$ is interesting for the classification program and boils down to finding a condition which ensures that S is an isomorphism. Dadarlat and Loring proved in [45, Thm. 4.3] that S is an isomorphism for all separable C^*-algebras B if and only if $[[\mathrm{id}_{A\otimes\mathbb{K}}]] \in [[A \otimes \mathbb{K}, A \otimes \mathbb{K}]]$ has an additive inverse. These C^*-algebras A are called homotopy symmetric. It is straightforward to check that $A \otimes \mathbb{K}$ can not contain non-zero projections if A is homotopy symmetric.

As elegant as the condition in [45] is, it is still quite hard to check. In chapter 6 we define property (QH), which is based on discrete asymptotic morphisms: A satisfies property (QH) if we have an injective discrete asymptotic morphism $\eta_n \colon A \to \mathbb{K}$, which is nullhomotopic

(Def. 6.2.6). This is equivalent to homotopy symmetry for *nuclear* C^*-algebras as we show in our the main theorem (Thm. 6.3.1). The proof uses the comparison of discrete and continuous asymptotic morphisms from [136].

Our new property is very closely related to approximate representations, which makes it easier to check. It also has very good permanence properties as we show in Thm. 6.3.3. One of these is a 2-out-of-3-property for split exact sequences. Using it we can prove that a separable continuous $C(X)$-algebra over a compact metrizable space X with nuclear fibers is homotopy symmetric if this is true for one of the fibers (Cor. 6.3.4).

Let G be a discrete countable torsion-free abelian group. We have $C^*G \cong C(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual of G. The kernel $I(G)$ of the trivial representation $C^*G \to \mathbb{C}$ corresponds to $C_0(\widehat{G} \setminus \{e\})$. In particular $I(G)$ is homotopy symmetric, which rises the question whether the same holds for all discrete countable torsion-free amenable groups. This was conjectured by Dadarlat in [44]. We prove this conjecture for discrete countable torsion-free nilpotent groups G in Thm. 6.4.3. The proof uses an induction over the length of the upper central series together with Thm. 6.4.1 based on Cor. 6.3.4.

publication status: The material in this chapter is contained in a joint paper with Marius Dadarlat and is submitted for publication, but still under review [46].

1.4.6 Crossed module actions on continuous trace C^*-algebras

As we have seen above, the basic setup for T-duality consists of a principal $\mathbb{T}^n$-bundle $P \to X$ over a locally compact Hausdorff space X together with a continuous trace C^*-algebra A with spectrum P. To construct a T-dual for A we have to lift the canonical $\mathbb{R}^n$-action on P to A. In the case $n = 1$ this always works in a unique way, but for higher dimensional tori there are topological obstructions, which arise from the fact that A may be non-trivial over the orbits of the action.

In case the action of $\mathbb{R}^n$ on P lifts to A, the crossed product $A' = A \rtimes \mathbb{R}^n$ is either again a continuous trace C^*-algebra or a continuous field of non-commutative tori [97, Thm. 3.1]. In the first case the spectrum P' of A' turns out to be again a principal $\mathbb{T}^n$-bundle over X and A' is a classical T-dual of A. If there is no lift, Bouwknegt, Hannabuss and Mathai suggested in [22] that non-associative C^*-algebras might play the role of the T-dual.

In the main theorems of this chapter we prove that even though the $\mathbb{R}^n$-action might not lift to A, a crossed module extension $\mathcal{H}$ of $\mathbb{R}^n$ always acts on A. Crossed modules represent strictifications of 2-groups, which are 2-categorical analogues of ordinary groups. The weak 2-group $\mathcal{G}$, which we consider, has a point as its object space, $\mathbb{R}^n$ as its space of 1-morphisms and the trivial $\Lambda^3\mathbb{R}^n$-bundle over $\mathbb{R}^n$ as its space of 2-morphisms. It has trivial unitors, but a non-trivial associator (Def. 7.2.2). If we strictify, we obtain the crossed module $\mathcal{H}$. Let

$\mathcal{G} \ltimes P$ and $\mathcal{H} \ltimes P$ be the associated transformation bigroupoids.

In Thm. 7.4.3 we construct a continuous action of $\mathcal{G} \ltimes P$ on A via correspondences as defined in section 7.3. In essence this means that we send $p \in P$ to the fiber A_p, a 1-morphism $t \in \mathbb{R}^n$ to a Morita bimodule between A_p and $A_{\alpha_t(p)}$ and the 2-morphisms to bimodule maps. All of this data has to satisfy certain consistency conditions. In Thm. 7.4.7 we note that we can strictify this action to one of $\mathcal{H} \ltimes P$.

We also define the equivariant Brauer group $\mathrm{Br}_{\mathcal{H}}(P)$ in our setting as equivalence classes of continuous trace C^*-algebras carrying an action of $\mathcal{H}$ (compare with [36]). Note that $\mathrm{Br}(P)$ and $\mathrm{Br}_{\mathcal{H}}(P)$ denote the full Brauer groups here, not just the torsion part as above. The group $\mathrm{Br}_{\mathcal{H}}(P)$ fits into an exact sequence as proven in Thm. 7.4.4. The upshot of this construction is that it combines all topological obstructions into one structure and it yields a non-associative Fell bundle, which encodes the T-dual in the same way as the non-associative algebras in [22].

publication status: The material in this chapter is contained in a joint paper with Ralf Meyer and is submitted for publication, but still under review [105].

2 A Dixmier-Douady theory for strongly self-absorbing C^*-algebras

We show that the Dixmier-Douady theory of continuous fields of C^*-algebras with compact operators $\mathbb{K}$ as fibers extends significantly to a more general theory of fields with fibers $A\otimes\mathbb{K}$ where A is a strongly self-absorbing C^*-algebra. The classification of the corresponding locally trivial fields involves a generalized cohomology theory which is computable via the Atiyah-Hirzebruch spectral sequence. An important feature of the general theory is the appearance of characteristic classes in higher dimensions. We also give a necessary and sufficient K-theoretical condition for local triviality of these continuous fields over spaces of finite covering dimension.

2.1 Introduction

Continuous fields of C^*-algebras are employed as versatile tools in several areas, including index and representation theory, the Novikov and the Baum-Connes conjectures [32], (deformation) quantization [117, 89] and the study of the quantum Hall effect [14]. While continuous fields play the role of bundles in C^*-algebra theory, the underlying bundle structure is typically not locally trivial. Nevertheless, these bundles have sufficient continuity properties to allow for local propagation of interesting K-theory invariants along their fibers. Continuous fields of C^*-algebras with simple fibers occur naturally as the class of C^*-algebras with Hausdorff primitive spectrum.

In a classic paper [55], Dixmier and Douady studied the continuous fields of C^*-algebras with fibers (stably) isomorphic to the compact operators $\mathbb{K}=\mathbb{K}(H)$ (H an infinite dimensional Hilbert space) over a paracompact base space X. In this article we develop a general theory of continuous fields with fibers $A\otimes\mathbb{K}$ where A is a strongly self-absorbing C^*-algebra. We show that the results of [55] fit naturally and admit significant generalizations in the new theory. The classification of these fields involves suitable generalized cohomology theories. An important feature of the new theory is the appearance of characteristic classes in higher dimensions.

As a byproduct of our approach we find an operator algebra realization of the classic spectrum $BBU_{\otimes}$. Let us recall that for a compact connected metric space X the invertible elements of the K-theory ring $K^0(X)$ is an abelian group $K^0(X)^{\times}$ whose elements are represented by classes of vector bundles of virtual rank ± 1, corresponding to homotopy classes $[X, \mathbb{Z}/2 \times BU_{\otimes}]$. The group operation is induced by the tensor product of vector bundles. Segal has shown that $BU_{\otimes}$ is in fact an infinite loop space and hence there is a cohomology theory $bu^*_{\otimes}(X)$ such that $K^0(X)^{\times}$ is just the 0-group $bu^0_{\otimes}(X)$ of this theory [129], but gave no geometric interpretation for the higher order groups. Our results lead to a geometric realization of the first group $bu^1_{\otimes}(X)$ as the isomorphism classes of locally trivial bundles of C^*-algebras with fiber the stabilized Cuntz algebra $\mathcal{O}_\infty \otimes \mathbb{K}$ where the group operation corresponds to the tensor product, see [49].

Let us recall two central results of Dixmier and Douady from [55].

Theorem 2.1.1. *The isomorphism classes of locally trivial fields over X with fibers $\mathbb{K}$ form a group under the operation of tensor product and this group is isomorphic to $\check{H}^3(X, \mathbb{Z})$.*

Theorem 2.1.2. *If X is finite dimensional, then a separable continuous field B over X with fibers isomorphic to $\mathbb{K}$ is locally trivial if and only if it satisfies Fell's condition, i.e. each point of X has a closed neighborhood V such that the restriction of B to V contains a projection of constant rank* 1.

The corresponding characteristic class $\delta(B) \in \check{H}^3(X, \mathbb{Z})$ is now known as the Dixmier-Douady invariant. Most prominent among its applications is its appearance as twisting class in twisted K-theory [56, 122, 8, 9], which is the natural home for D-brane charges in string theory [130, 23]. A recent friendly introduction to the Dixmier-Douady theory can be found in [125].

The class of strongly self-absorbing C^*-algebras, introduced by Toms and Winter [139], is closed under tensor products and contains C^*-algebras that are cornerstones of Elliott's classification program of simple nuclear C^*-algebras: the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang-Su algebra $\mathcal{Z}$, the canonical anticommutation relations algebra M_{2^∞} and in fact all UHF-algebras of infinite type. These are separable C^*-algebras singled out by a crucial property: there exists an isomorphism $A \to A \otimes A$, which is unitarily homotopic to the map $a \mapsto a \otimes 1_A$, [51, 150]. Using this property, which is equivalent to, but formally much stronger than the original definition of [139], we prove that

- $\mathrm{Aut}(A)$ is contractible in the point-norm topology.
- $\mathrm{Aut}(A \otimes \mathbb{K})$ is well-pointed and it has the homotopy type of a CW-complex.
- The classifying space $B\mathrm{Aut}(A \otimes \mathbb{K})$ of locally trivial C^*-algebra bundles with fiber $A \otimes \mathbb{K}$ carries an H-space structure induced by the tensor product. Moreover, this tensor product multiplication is homotopy commutative up to all higher homotopies

and therefore equips $B\mathrm{Aut}(A \otimes \mathbb{K})$ with the structure of an infinite loop space by results of Segal and May.

These properties mirror entirely the corresponding properties of $\mathrm{Aut}(\mathbb{K}) = PU(H)$ and $B\mathrm{Aut}(\mathbb{K}) = BPU(H)$ obtained by their identification with the Eilenberg-MacLane spaces $K(\mathbb{Z}, 2)$ and respectively $K(\mathbb{Z}, 3)$ which we implicitly reprove as they correspond to the case $A = \mathbb{C}$. Recall that if X is paracompact Hausdorff, then $\check{H}^n(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)]$, [75].

It is worth noting that while the obstructions to having a natural group structure on the isomorphism classes of locally trivial continuous fields with fiber $A \otimes \mathbb{K}$ – such as nontrivial Samelson products [108, Sec.6] – do vanish in the strongly self-absorbing case, that is not necessarily true for general self-absorbing C^*-algebras, i.e. those with $A \otimes A \cong A$. This motivates yet again our choice of fibers. In complete analogy with Theorem 2.1.1 we have:

Theorem A. *Let X be a compact metrizable space and let A be a strongly self-absorbing C^*-algebra. The set $\mathcal{B}un_X(A \otimes \mathbb{K})$ of isomorphism classes of locally trivial fields over X with fiber $A \otimes \mathbb{K}$ becomes an abelian group under the operation of tensor product. Moreover, this group is isomorphic to $E^1_A(X)$, the first group of a generalized connective cohomology theory $E^*_A(X)$ defined by the infinite loop space $B\mathrm{Aut}(A \otimes \mathbb{K})$.*

We also show that the zero group $E^0_A(X)$ computes the homotopy classes $[X, \mathrm{Aut}(A \otimes \mathbb{K})]$ and it is isomorphic to the group of positive invertible elements of the abelian ring $K_0(C(X) \otimes A)$, denoted by $K_0(C(X) \otimes A)^\times_+$, for $A \neq \mathbb{C}$. In particular, we fully compute the coefficients of $E^*_A(X)$, as they are given by the homotopy groups

$$\pi_i(\mathrm{Aut}(A \otimes \mathbb{K})) = \begin{cases} K_0(A)^\times_+ & \text{if } i = 0 \\ K_i(A) & \text{if } i \geq 1 \ . \end{cases}$$

$K_0(A)$ has a natural ring structure with unit given by the class of 1_A. $K_0(A)^\times$ denotes the group of multiplicative elements of $K_0(A)$ and $K_0(A)^\times_+$ is its subgroup consisting of positive elements.

The Atiyah-Hirzebruch spectral sequence then allows us to obtain classification results for locally trivial $A \otimes \mathbb{K}$-bundles over X. In the case of the universal UHF algebra $M_\mathbb{Q}$, bundles with fiber $M_\mathbb{Q} \otimes \mathbb{K}$ are essentially classified by the ordinary rational cohomology groups of odd degree of the underlying space:

$$\mathcal{B}un_X(M_\mathbb{Q} \otimes \mathbb{K}) \cong E^1_{M_\mathbb{Q}}(X) \cong H^1(X, \mathbb{Q}^\times_+) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}).$$

A similar result holds for bundles with fiber $\mathcal{O}_\infty \otimes M_\mathbb{Q} \otimes \mathbb{K}$, see Corollary 2.4.5. It follows that if A is any strongly self-absorbing C^*-algebra that satisfies the UCT, then there are rational characteristic classes $\delta_k : \mathcal{B}un_X(A \otimes \mathbb{K}) \to H^{2k+1}(X, \mathbb{Q})$ such that $\delta_k(B_1 \otimes B_2) =$

$\delta_k(B_1) + \delta_k(B_2)$.

An unexpected consequence of our results is that for any strongly self-absorbing C^*-algebra A, if two bundles $B_1, B_2 \in \mathcal{B}un_X(A \otimes \mathbb{K})$ become isomorphic after tensoring with $\mathcal{O}_\infty$, then they must be isomorphic in the first place, see Corollary 2.4.9.

Our result concerning local triviality is the following generalization of Theorem 2.1.2 which involves a K-theoretic interpretation of Fell's condition.

Theorem B. *Let X be a locally compact metrizable space of finite covering dimension and let A be a strongly self-absorbing C^*-algebra. A separable continuous field B over X with fibers abstractly isomorphic to $A \otimes \mathbb{K}$ is locally trivial if and only if for each point $x \in X$, there exist a closed neighborhood V of x and a projection $p \in B(V)$ such that $[p(v)] \in K_0(B(v))^\times$ for all $v \in V$.*

A notable consequence of Theorem B is that any separable continuous field of C^*-algebras over X with all fibers abstractly isomorphic to $M_\mathbb{Q} \otimes \mathbb{K}$ is locally trivial and therefore, by Theorem A, it is determined up to isomorphism by its class in $E^1_{M_\mathbb{Q}}(X) \cong H^1(X, \mathbb{Q}^\times_+) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$.

The condition that X is finite dimensional is essential in Theorem B, as shown by examples constructed in [55] for $A = \mathbb{C}$, [72] for $A = M_\mathbb{Q}$ and [43] for $A = \mathcal{O}_2$.

Let us recall that a C^*-algebra isomorphic to the compact operators on a separable (possibly finite dimensional) Hilbert space is called an elementary C^*-algebra. Dixmier and Douady gave two other results concerning continuous fields of elementary C^*-algebras:

(i) If B is a continuous field of elementary C^*-algebras that satisfies Fell's condition, then $B \otimes \mathbb{K}$ is locally trivial.

(ii) The class $\delta(B) \in \check{H}^3(X, \mathbb{Z})$ can be defined for any continuous field of elementary C^*-algebras that satisfies Fell's condition. Moreover B is isomorphic to the compact operators of a continuous field of Hilbert spaces if and only if $\delta(B) = 0$.

We extend (i) and the first part of (ii) to general strongly self-absorbing C^*-algebras, but we must require finite dimensionality for either the fiber or the base space in order to obtain a perfect analogy with these results, see Corollaries 2.4.10 and 2.4.11. These restrictions are necessary. Indeed, while any unital separable continuous field of C^*-algebras with fiber $\mathbb{C}$ over X is locally trivial (in fact isomorphic to $C_0(X)$), automatic local triviality fails if $\mathbb{C}$ is replaced by strongly self-absorbing C^*-algebras such as $M_\mathbb{Q}$ and $\mathcal{O}_2$, see [72] and [43].

This fact also explains why the second part of (ii) is specific to fields of elementary C^*-algebras. Our set-up allows us to associate rational characteristic classes to any continuous fields (satisfying a weak Fell's condition) whose fibers are Morita equivalent to strongly self-absorbing C^*-algebras which are not necessarily mutually isomorphic. Such fields are typically very far from being locally trivial. We refer the reader to Section 2.4 for further discussion.

The homotopy equivalence $\mathrm{Aut}(A \otimes \mathbb{K}) \simeq K_0(A)_+^\times \times BU(A)$ (see Corollary 2.2.17) suggests that the generalized cohomology theory associated to $\mathrm{Aut}(A \otimes \mathbb{K})$ is very closely related to the unit spectrum $GL_1(KU^A)$ of topological K-theory with coefficients in the group $K_0(A)$. This is again parallel to the Dixmier-Douady theory, where we have $\mathrm{Aut}(\mathbb{K}) = PU(H) \simeq BU(1) \subset GL_1(KU)$. We will make this connection precise in [49]. Let us just mention here that the homotopy equivalence $\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K}) \simeq BU$ deloops to a homotopy equivalence $B\mathrm{Aut}(\mathcal{Z}\otimes\mathbb{K}) \simeq B(BU_\otimes)$. This unveils a very natural operator algebra realization of the classic Ω-spectrum $B(BU_\otimes)$ introduced by Segal [129].

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2.2 The topology of $\mathrm{Aut}(A \otimes \mathbb{K})$

The automorphism group $\mathrm{Aut}(B)$ of a separable C^*-algebra B, equipped with the point-norm topology, is a separable and metrizable topological group. In particular its topology is compactly generated. We are going to show in this section that if A is a strongly self-absorbing C^*-algebra, then $\mathrm{Aut}(A \otimes \mathbb{K})$ is well-pointed and has the homotopy type of a CW-complex. This will enable us to apply the standard techniques of algebraic topology, in particular when it comes to dealing with its classifying space. We denote by $\simeq$ the relation of homotopy equivalence.

2.2.1 Strongly self-absorbing C^*-algebras

Let us recall from [139] that a C^*-algebra A is strongly self-absorbing if it is separable, unital and there exists a $*$-isomorphism $\psi : A \to A \otimes A$ such that ψ is approximately unitarily equivalent to the map $l : A \to A \otimes A$, $l(a) = a \otimes 1_A$. It follows from [51] and [150] that ψ and l must be in fact unitarily homotopy equivalent, see Theorem 2.2.1(b). Note that, unlike [139], we don't exclude the complex numbers $\mathbb{C}$ from the class of strongly self-absorbing C^*-algebras. For future reference, we collect under one roof an important series of results due to several authors.

Theorem 2.2.1. *A strongly self-absorbing C^*-algebra A has the following properties:*

(a) A is simple, nuclear and is either stably finite or purely infinite; if it is stably finite, then it admits a unique trace, see [139] and references therein.

(b) Let B be a unital separable C^*-algebra. For any two unital $*$-homomorphisms $\alpha, \beta : A \to B \otimes A$ there is a continuous path of unitaries $(u_t)_{t\in[0,1)}$ in $B \otimes A$ such that $u_0 = 1$ and $\lim_{t\to 1} \|u_t \alpha(a) u_t^* - \beta(a)\| = 0$ for all $a \in A$. This property was proved in

[51, Thm.2.2] under the assumption that A is K_1-injective. Winter [150] has shown that any infinite dimensional strongly self-absorbing C^*-algebra A is $\mathcal{Z}$-stable, i.e. $A \otimes \mathcal{Z} \cong A$, and hence A is K_1-injective by a result of Rørdam [120].

(c) Any unital $\mathcal{Z}$-stable C^*-algebra has cancellation of full projections by a result of Jiang [77, Thm.1]. In particular, if B is a separable unital C^*-algebra and $A \neq \mathbb{C}$, then $B \otimes A$ is isomorphic to $B \otimes A \otimes \mathcal{Z}$ and hence it has cancellation of full projections.

(d) If B is a unital $\mathcal{Z}$-stable C^*-algebra, then $\pi_0(U(B)) \cong K_1(B)$, by [77, Thm.2].

(e) If A satisfies the Universal Coefficient Theorem (UCT) in KK-theory, then $K_1(A) = 0$ by [139]. If in addition A is purely infinite, then A is isomorphic to either $\mathcal{O}_2$ or $\mathcal{O}_\infty$ or a tensor product of $\mathcal{O}_\infty$ with a UHF-algebra of infinite type [139, Cor.5.2].

Notation. For C^*-algebras A, B we denote by $\mathrm{Hom}(A, B)$ the space of full $*$-homomorphisms from A to B and by $\mathrm{End}(A)$ the space of full $*$-endomorphisms of A. Recall that a $*$-homomorphism $\varphi : A \to B$ is full if for any nonzero element $a \in A$, the closed ideal generated by $\varphi(a)$ is equal to B. If A is a unital C^*-algebra, we denote by $K_0(A)_+$ the subsemigroup of $K_0(A)$ consisting of classes $[p]$ of full projections $p \in A \otimes \mathbb{K}$.

2.2.2 Contractibility of $\mathrm{Aut}(A)$

While it is known from [51, Cor.3.1] that $\mathrm{Aut}(A)$ is weakly contractible in the point norm-topology, we can strengthen this result by combining it with the idea of half-flips from [139].

Let B be a separable C^*-algebra and let $e \in B$ be a projection. Consider the following spaces of $*$-endomorphisms of B endowed with the point-norm topology.

$$\mathrm{End}_e(B) = \{\alpha \in \mathrm{End}(B) \colon \ \alpha(e) = e\}, \quad \mathrm{Aut}_e(B) = \{\alpha \in \mathrm{Aut}(B) \colon \ \alpha(e) = e\}.$$

Let $l, r : B \to B \otimes B$ (minimal tensor product) be defined by $l(b) = b \otimes e$ and $r(b) = e \otimes b$.

Lemma 2.2.2. *Suppose that there is a continuous map $\psi : [0,1] \to \mathrm{Hom}(B, B \otimes B)$ such that $\psi(0) = l$, $\psi(1) = r$, $\psi(t)(e) = e \otimes e$ and $\psi(t)$ is a $*$-isomorphism for all $t \in (0,1)$. Then $\mathrm{Aut}_e(B)$ and $\mathrm{End}_e(B)$ are contractible spaces.*

Proof. First we deal with $\mathrm{Aut}_e(B)$. Consider $H \colon I \times \mathrm{Aut}_e(B) \to \mathrm{Aut}_e(B)$ defined by

$$H(t, \alpha) = \begin{cases} \alpha & \text{for } t = 0 \\ \psi(t)^{-1} \circ (\alpha \otimes \mathrm{id}_B) \circ \psi(t) & \text{for } 0 < t < 1 \ , \\ \mathrm{id}_B & \text{for } t = 1 \ . \end{cases} \tag{2.1}$$

Note that $H(t,\alpha)(e) = e$ since $\psi(t)(e) = e \otimes e$. Observe that $(\alpha \otimes \mathrm{id}_B) \circ l = l \circ \alpha$. It is straightforward to verify the continuity of H at points (α, t) with $t \neq 0$ and $t \neq 1$. Let $b \in B$, let $t_n \in (0,1)$ be a net converging to 0 and let $\alpha_i \in \mathrm{Aut}_e(B)$ be a net converging to $\alpha \in \mathrm{Aut}_e(B)$. The estimate,

$$\begin{aligned} & \|(\psi(t_n)^{-1} \circ (\alpha_i \otimes \mathrm{id}_B) \circ \psi(t_n))(b) - \alpha(b)\| = \|(\alpha_i \otimes \mathrm{id}_B) \circ \psi(t_n)(b) - \psi(t_n) \circ \alpha(b)\| \\ \leq \; & \|(\alpha_i \otimes \mathrm{id}_B) \circ \psi(t_n)(b) - (\alpha_i \otimes \mathrm{id}_B) \circ l(b)\| + \|(\alpha_i \otimes \mathrm{id}_B) \circ l(b) - (\alpha \otimes \mathrm{id}_B) \circ l(b)\| \\ + \; & \|(\alpha \otimes \mathrm{id}_B) \circ l(b) - \psi(t_n) \circ \alpha(b)\| \\ \leq \; & \|\psi(t_n)(b) - l(b)\| + \|\alpha_i(b) - \alpha(b)\| + \|(\alpha \otimes \mathrm{id}_B) \circ l(b) - \psi(t_n) \circ \alpha(b)\| \end{aligned}$$

implies the continuity of H at $(\alpha, 0)$. An analogous argument using $(\alpha \otimes \mathrm{id}_B) \circ r = r$ shows continuity at $(\alpha, 1)$. We also have $H(t, \mathrm{id}_B) = \mathrm{id}_B$ for all $t \in [0,1]$. Thus, H provides a (strong) deformation retraction of $\mathrm{Aut}_e(B)$ to id_B. The argument for the contractibility of $\mathrm{End}_e(B)$ is entirely similar. One observes that equation (2.1) also defines a map $H \colon I \times \mathrm{End}_e(B) \to \mathrm{End}_e(B)$ which gives a deformation retraction of $\mathrm{End}_e(B)$ to id_B.. □

Theorem 2.2.3. *Let A be a strongly self-absorbing C^*-algebra. Then $\mathrm{Aut}(A)$ and $\mathrm{End}_{1_A}(A)$ are contractible spaces.*

Proof. Let $l, r \colon A \to A \otimes A$ be the maps $l(a) = a \otimes 1_A$ and $r(a) = 1_A \otimes a$. Fix an isomorphism $\psi \colon A \to A \otimes A$. It follows from Theorem 2.2.1(b) that there exists a continuous path of unitaries $u \colon (0,1] \to U(A \otimes A)$ with $u(1) = 1_{A \otimes A}$ such that

$$\lim_{t \to 0} \|u(t)\, \psi(a)\, u(t)^* - l(a)\| = 0 \ .$$

Define $\psi_l \colon (0,1] \to \mathrm{Iso}(A, A \otimes A)$ by $\psi_l(t) = \mathrm{Ad}_{u(t)} \circ \psi$. Likewise there is a continuous path of unitaries $v \colon [0,1) \to U(A \otimes A)$ with $v(0) = 1_{A \otimes A}$ and such that

$$\lim_{t \to 1} \|v(t)\, \psi(a)\, v(t)^* - r(a)\| = 0 \ .$$

Define $\psi_r \colon [0,1) \to \mathrm{Iso}(A, A \otimes A)$ by $\psi_r(t) = \mathrm{Ad}_{v(t)} \circ \psi$. By juxtaposing the paths ψ_l and ψ_r we obtain a homotopy from l to r which satisfies the assumptions of Lemma 2.2.2 with $e = 1_A$. It follows that $\mathrm{Aut}(A)$ and $\mathrm{End}_{1_A}(A)$ are contractible spaces. □

The following is a minor variation of a result of Blackadar [18, p.57] and Herman and Rosenberg [68].

Lemma 2.2.4. *Let A and B be separable AF-algebras and let $e \in A$ be a projection. Suppose that $\varphi, \psi : A \to B$ are two $*$-homomorphisms such that $\varphi(e) = \psi(e)$ and $\varphi_* = \psi_* : K_0(A) \to K_0(B)$. Then there is a continuous map $u : [0,1) \to U(B^+)$ with $u(0) = 1$, $[u(t), \psi(e)] = 0$ for all $t \in [0,1)$ and such that $\lim_{t \to 1} \|u(t)\psi(a)u(t)^* - \varphi(a)\| = 0$ for all $a \in A$.*

Proof. If B is a nonunital C^*-algebra, we regard B as a C^*-subalgebra of its unitization B^+. Write A as the closure of an increasing union of finite dimensional C^*-subalgebras $A_n \subset A_{n+1}$ with $A_0 = \mathbb{C}e$. Since $\phi_* = \psi_*$, for each $n \geq 0$ we find a unitary $u_n \in U(B^+)$ such that $u_n \psi(x) u_n^* = \varphi(x)$ for all $x \in A_n$ and $u_0 = 1$. Observe that $w_n = u_{n+1}^* u_n$ is a unitary in the commutant C_n of $\psi(A_n)$ in B^+. This commutant is known to be an AF-algebra, see [68, Lemma 3.1]. Therefore there is a continuous path of unitaries $t \mapsto W_n(t) \in U(C_n)$, $t \in [n, n+1]$, such that $W_n(n) = w_n$ and $W_n(n+1) = 1$. Define a continuous map $u : [0,\infty) \to U(B)$ by $u(t) = u_{n+1} W_n(t)$, $t \in [n, n+1]$. One verifies immediately that $[u(t), \psi(e)] = 0$ for all t and that $u(t)\psi(x)u(t)^* = \varphi(x)$ for all $x \in A_n$ and $t \in [n, n+1]$. It follows that $\lim_{t\to\infty} \|u(t)\psi(a)u(t)^* - \varphi(a)\| = 0$ for all $a \in A$. □

Theorem 2.2.5. *Let A be a strongly self-absorbing C^*-algebra and let $e \in \mathbb{K}$ be a rank-1 projection. Then the stabilizer group* $\mathrm{Aut}_{1\otimes e}(A \otimes \mathbb{K})$ *and the space* $\mathrm{End}_{1\otimes e}(A \otimes \mathbb{K})$ *are contractible.*

Proof. We shall use the following consequence of Lemma 2.2.4. Let $\varphi_0, \varphi_1 : \mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$ be two $*$-homomorphisms such that $\varphi_0(e) = \varphi_1(e) = e \otimes e$. Fix a $*$-isomorphism $\psi_{1/2} : \mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$ with $\psi_{1/2}(e) = e \otimes e$. By applying Lemma 2.2.4 to both pairs $(\varphi_i, \psi_{1/2})$, $i = 0, 1$, we find a continuous map $\psi : [0,1] \to \mathrm{Hom}(\mathbb{K}, \mathbb{K} \otimes \mathbb{K})$ such that $\psi(0) = \varphi_0$, $\psi(1) = \varphi_1$, $\psi(t)(e) = e \otimes e$ and $\psi(t)$ is a $*$-isomorphism for all $t \in (0,1)$.

We proceed in much the same way as the proof of Theorem 2.2.3, by an application Lemma 2.2.2. Let $l, r \colon A \to A \otimes A$ be defined by $l(a) = a \otimes 1_A$ and $r(a) = 1_A \otimes a$. We have seen in the proof of Theorem 2.2.3 that there is a continuous map $\psi : [0,1] \to \mathrm{Hom}(A, A \otimes A)$ such that $\psi(0) = l$, $\psi(1) = r$, and $\psi(t)$ is a $*$-isomorphism for all $t \in (0,1)$.

Let $l_{\mathbb{K}}, r_{\mathbb{K}} \colon \mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$ be given by $l_{\mathbb{K}}(x) = x \otimes e$, $r_{\mathbb{K}}(x) = e \otimes x$. Using the remark from the beginning of the proof, we find a continuous map $\psi_{\mathbb{K}} : [0,1] \to \mathrm{Hom}(\mathbb{K}, \mathbb{K} \otimes \mathbb{K})$ such that $\psi_{\mathbb{K}}(0) = l_{\mathbb{K}}$, $\psi_{\mathbb{K}}(1) = r_{\mathbb{K}}$, $\psi_{\mathbb{K}}(t)(e) = e \otimes e$ and $\psi_{\mathbb{K}}(t)$ is a $*$-isomorphism for all $t \in (0,1)$.

Let $A_{\mathbb{K}} = A \otimes \mathbb{K}$ and consider the $*$-homomorphisms $\hat{l}$, $\hat{r} \colon A_{\mathbb{K}} \to A_{\mathbb{K}} \otimes A_{\mathbb{K}}$ with

$$\hat{l} = \sigma \circ (l \otimes l_{\mathbb{K}}) \quad \text{and} \quad \hat{r} = \sigma \circ (r \otimes r_{\mathbb{K}}) ,$$

where $\sigma \colon A \otimes (A \otimes \mathbb{K}) \otimes \mathbb{K} \to (A \otimes \mathbb{K}) \otimes (A \otimes \mathbb{K})$ interchanges the second and third tensor factor. Note that $\hat{l}(a \otimes x) = (a \otimes x) \otimes (1_A \otimes e)$ and $\hat{r}(a \otimes x) = (1_A \otimes e) \otimes (a \otimes x)$ for $a \in A$ and $x \in \mathbb{K}$. Define $\hat{\psi} : [0,1] \to \mathrm{Hom}(A_{\mathbb{K}}, A_{\mathbb{K}} \otimes A_{\mathbb{K}})$ by $\hat{\psi} = \sigma \circ (\psi \otimes \psi_{\mathbb{K}})$. Then $\hat{\psi}(0) = \hat{l}$, $\hat{\psi}(1) = \hat{r}$, $\hat{\psi}(t)(1_A \otimes e) = (1_A \otimes e) \otimes (1_A \otimes e)$ and $\hat{\psi}(t)$ is an isomorphism for all $t \in (0,1)$. It follows by Lemma 2.2.2 that $\mathrm{Aut}_{1\otimes e}(A \otimes \mathbb{K})$ and $\mathrm{End}_{1\otimes e}(A \otimes \mathbb{K})$ are contractible. □

Remark 2.2.6. Taking $A = \mathbb{C}$, Thm. 2.2.5 reproves the contractibility of $U(H)$ in the strong topology.

2.2.3 The homotopy type of $\mathrm{Aut}_0(A \otimes \mathbb{K})$

For a C^*-algebra B we denote by $\mathrm{Aut}_0(B)$ and $\mathrm{End}_0(B)$ the path-connected component of the identity. We have seen in Theorem 2.2.3 that for a strongly self-absorbing C^*-algebra A the space $\mathrm{Aut}(A)$ is contractible. In particular, it has the homotopy type of a CW-complex. In this section, we will extend the latter statement to the space $\mathrm{Aut}_0(A \otimes \mathbb{K})$, which is no longer contractible, but has a very interesting homotopy type. We start by considering the subspace of projections in $A \otimes \mathbb{K}$, denoted by $\mathcal{P}r(A \otimes \mathbb{K})$.

Lemma 2.2.7. *Let B be a C^*-algebra. The space $\mathcal{P}r(B)$ has the homotopy type of a CW-complex.*

Proof. Let B_{sa} be the real Banach space of self-adjoint elements in B. Consider the subset U of B_{sa} consisting of all elements which do not have $1/2$ in the spectrum. Since invertibility is an open condition, U is an open subset of B_{sa} and therefore has the homotopy type of a CW-complex by [93, Cor.5.5, p.134]. Since $\sigma(p) \subset \{0, 1\}$ for any projection $p \in B$, we have $\mathcal{P}r(B) \subset U$. Let f be the characteristic function of the interval $(\frac{1}{2}, \infty)$. By functional calculus, f induces a continuous map $U \to \mathcal{P}r(B)$, $a \mapsto f(a)$, which restricts to the identity on $\mathcal{P}r(B)$. Thus, $\mathcal{P}r(B)$ is dominated by a space having the homotopy type of a CW-complex. By [93, Cor.3.9, p.127] it is homotopy equivalent to a CW-complex itself. $\square$

Let e be a rank-1 projection in $\mathbb{K}$. We define $\mathcal{P}r_0(A \otimes \mathbb{K})$ to be the connected component of $1 \otimes e \in \mathcal{P}r(A \otimes \mathbb{K})$. It does not depend on the choice of e as long as the rank of e is equal to 1.

Lemma 2.2.8. *Let A be a unital C^*-algebra and let $e \in \mathbb{K}$ be a rank-1 projection. Then the maps $\mathrm{Aut}_0(A \otimes \mathbb{K}) \to \mathcal{P}r_0(A \otimes \mathbb{K})$ and $\mathrm{End}_0(A \otimes \mathbb{K}) \to \mathcal{P}r_0(A \otimes \mathbb{K})$ which send α to $\alpha(1 \otimes e)$ are locally trivial fiber bundles over a paracompact base space and therefore Hurewicz fibrations.*

Proof. This is a particular case of a more general result, which we will prove for $\mathrm{End}_0(A \otimes \mathbb{K})$. The proof for the sequence of automorphism groups is entirely analogous. Let B be a C^*-algebra, let $q \in \mathcal{P}r(B)$ and let $\mathcal{P}r_0(B)$ be the path-component of q. Then $\pi : \mathrm{End}_0(B) \to \mathcal{P}r_0(B)$, $\pi(\alpha) = \alpha(q)$ is in fact a locally trivial bundle with fiber $\mathrm{End}_q(B)$. The map π is well-defined. Indeed, if α is homotopic to id_B, then the projection $\alpha(q)$ is connected to q by a continuous path in $\mathcal{P}r(B)$.

Let $U_0(B^+)$ denote the path-component of 1 in the unitary group of the unitization of B. Thus, for $u \in U_0(B^+)$ we have $\mathrm{Ad}_u \in \mathrm{Aut}_0(B) \subseteq \mathrm{End}_0(B)$. By definition any $p \in \mathcal{P}r_0(B)$ is homotopic to q. Therefore p and q are also unitarily equivalent via a unitary $u \in U_0(B^+)$. Since $\pi(\mathrm{Ad}_u) = p$ it follows that π is surjective. Let $p_0 \in \mathcal{P}r_0(B)$ and let U be its the open neighborhood given by $U = \{p \in \mathcal{P}r_0(B) \mid \|p - p_0\| < 1\}$. If $p \in U$, then

$x_p = p_0p + (1 - p_0)(1 - p)$ is an invertible element of B^+. It follows that $u_p = x_p(x_p^*x_p)^{-\frac{1}{2}}$ is a unitary in $U_0(B^+)$ and the map $p \mapsto u_p$ is continuous with respect to the norm topologies [17, Prop.II.3.3.4]. Choose a unitary $v \in U_0(B^+)$ such that $p_0 = vqv^*$. Then $\sigma_{p_0} \colon U \to \mathrm{Aut}_0(B)$, $p \mapsto \mathrm{Ad}_{u_p^*v}$ is a continuous section of π over U and $\kappa_U \colon U \times \mathrm{End}_q(B) \to \mathrm{End}_0(B)$ defined by $\kappa_U(x, \beta) = \sigma_{p_0}(x) \circ \beta$ is a local trivialization with inverse $\tau_U \colon \mathrm{End}_0(B) \to U \times \mathrm{End}_q(B)$ given by $\tau_U(\beta) = (\beta(q), \sigma_{p_0}(\beta(q))^{-1} \circ \beta)$. This completes the proof. □

Corollary 2.2.9. *Let A be a strongly self-absorbing C^*-algebra. Then the spaces $\mathrm{Aut}_0(A \otimes \mathbb{K})$ and $\mathrm{End}_0(A \otimes \mathbb{K})$ both have the homotopy type of a CW-complex; they are homotopy equivalent to $\mathcal{P}r_0(A \otimes \mathbb{K}) \simeq BU(A)$.*

Proof. By Lemma 2.2.8 and Theorem 2.2.5 the spaces $\mathrm{Aut}_0(A \otimes \mathbb{K})$ and $\mathrm{End}_0(A \otimes \mathbb{K})$ are total spaces of fibrations where both base and fiber have the homotopy type of a CW-complex. Now the statement follows from [126, Thm.2], Theorem 2.2.5, except that it remains to argue that $\mathcal{P}r_0(A \otimes \mathbb{K}) \simeq BU(A)$. This is certainly known. The group $U(M(A \otimes \mathbb{K}))$ acts continuously and transitively on $\mathcal{P}r_0(A \otimes \mathbb{K})$ via $u \mapsto u(1 \otimes e)u^*$ with stabilizer $U(A) \times U(M(A \otimes \mathbb{K}))$. By the contractibility of $U(M(A \otimes \mathbb{K}))$ [37], $U(A) \to U(M(A \otimes \mathbb{K}))/1_A \times U(M(A \otimes \mathbb{K})) \to \mathcal{P}r_0(A \otimes \mathbb{K})$ is the *universal* principal $U(A)$-bundle. One uses the map $p \mapsto u_p$ constructed in the proof of Lemma 2.2.8 in order to verify local triviality. Thus $\mathcal{P}r_0(A \otimes \mathbb{K})$ is a model for $BU(A)$. □

2.2.4 The homotopy type of $\mathrm{Aut}(A \otimes \mathbb{K})$

In this section we compute the homotopy classes $[X, \mathrm{End}(A \otimes \mathbb{K})]$ and $[X, \mathrm{Aut}(A \otimes \mathbb{K})]$ in the case of a strongly self-absorbing C^*-algebra A and a compact metrizable space X, see Theorem 2.2.22. A similar topic was studied for Kirchberg algebras in [42]. Throughout this subsection e is a rank-1 projection in $\mathbb{K}$. Given a unital ring R, we denote by $R^\times$ the group of units in R. It is easily seen that $K_0(C(X) \otimes A)$ carries a ring structure with multiplication induced by an isomorphism $\psi : A \otimes \mathbb{K} \to A \otimes \mathbb{K} \otimes A \otimes \mathbb{K}$ which maps $1_A \otimes e$ to $1_A \otimes e \otimes 1_A \otimes e$. This structure does not depend on the choice ψ by Theorem 2.2.5. Let $\mathrm{End}(A \otimes \mathbb{K})^\times = \{\beta \in \mathrm{End}(A \otimes \mathbb{K}) \mid \beta(1 \otimes e) \text{ invertible in } K_0(A)\}$. We identify the space of continuous maps from X to $\mathrm{End}(A \otimes \mathbb{K})$ with $\mathrm{Hom}(A \otimes \mathbb{K}, C(X) \otimes A \otimes \mathbb{K})$ and with $\mathrm{End}_{C(X)}(C(X) \otimes A \otimes \mathbb{K})$. Similarly, we will identify the space of continuous maps from X to $\mathrm{Aut}(A \otimes \mathbb{K})$ with $\mathrm{Aut}_{C(X)}(C(X) \otimes A \otimes \mathbb{K})$.

Lemma 2.2.10. *Let A and B be unital separable C^*-algebras. Suppose that $p \in B \otimes \mathbb{K}$ is a full projection such that there is a unital $*$-homomorphism $\theta : A \to p(B \otimes \mathbb{K})p$. Then there is a $*$-homomorphism $\varphi : A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ such that $\varphi(1 \otimes e) = p$. If θ is an isomorphism, then we can choose φ to be an isomorphism.*

Proof. We denote by $\sim$ Murray-von Neumann equivalence of projections. Let us recall that if $q, r \in B \otimes \mathbb{K}$ are projections, then $q \sim r$ if and only if there is $u \in U(M(B \otimes \mathbb{K}))$ such that $uqu^* = r$, [106, Lemma 1.10]. Since p is a full projection in $B \otimes \mathbb{K}$, by [24], there is $v \in M(B \otimes \mathbb{K} \otimes \mathbb{K})$ such that $v^*v = p \otimes I$ and $vv^* = 1 \otimes I \otimes I$. Then $\gamma : p(B \otimes \mathbb{K})p \otimes \mathbb{K} \to B \otimes \mathbb{K} \otimes \mathbb{K}$, $\gamma(a) = vav^*$, is an isomorphism with the property that $\gamma(p \otimes e) = v(p \otimes e)v^* \sim (p \otimes e)(v^*v)(p \otimes e) = p \otimes e$. The map $\mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$, $x \mapsto x \otimes e$ is homotopic to a $*$-isomorphism as observed in the proof of Theorem 2.2.5. It follows that the map $B \otimes \mathbb{K} \to B \otimes \mathbb{K} \otimes \mathbb{K}$, $b \otimes x \mapsto b \otimes x \otimes e$ is also homotopic to a $*$-isomorphism μ. Note that $\mu(p) \sim p \otimes e \sim \gamma(p \otimes e)$. Thus, after conjugating μ by a unitary in $M(B \otimes \mathbb{K})$ we may arrange that $\mu(p) = \gamma(p \otimes e)$. It follows that $\varphi = \mu^{-1} \circ \gamma \circ (\theta \otimes \mathrm{id}_{\mathbb{K}}) \in \mathrm{Hom}(A \otimes \mathbb{K}, B \otimes \mathbb{K})$ has the property that $\varphi(1 \otimes e) = p$. Finally note that if θ is an isomorphism then so is φ. $\square$

Corollary 2.2.11 (Kodaka, [86]). *Let A be a separable unital C^*-algebra and let $p \in A \otimes \mathbb{K}$ be a full projection. Then $p(A \otimes \mathbb{K})p \cong A$ if and only if there is $\alpha \in \mathrm{Aut}(A \otimes \mathbb{K})$ such that $\alpha(1 \otimes e) = p$.*

Proposition 2.2.12. *Let A be a strongly self-absorbing C^*-algebra and let B be a separable unital C^*-algebra such that $B \cong B \otimes A$. Let $\varphi, \psi : A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ be two full $*$-homomorphisms. Suppose that $[\varphi(1_A \otimes e)] = [\psi(1_A \otimes e)] \in K_0(B)$. Then (i) φ is homotopic to ψ and (ii) φ is approximately unitarily equivalent to ψ, written $\varphi \approx_u \psi$.*

Proof. (i) For C^*-algebras A, B we denote by $[A, B]_\sharp$ the homotopy classes of full $*$-homomorphisms $\varphi : A \to B$. The inclusion $A \cong A \otimes e \hookrightarrow A \otimes \mathbb{K}$ induces a restriction map $\rho : [A \otimes \mathbb{K}, B \otimes \mathbb{K}]_\sharp \to [A, B \otimes \mathbb{K}]_\sharp$. Thomsen showed that ρ is bijective, see [135, Lemma 1.4]. Since the map $[\varphi] \mapsto [\varphi(1 \otimes e)]$ factors through ρ, it suffices to show that the map $[A, B \otimes \mathbb{K}]_\sharp \to K_0(B)$, $\varphi \mapsto [\varphi(1)]$ is injective. Let $\varphi, \psi : A \to B \otimes \mathbb{K}$ be two full $*$-homomorphisms. Suppose that $[\varphi(1)] = [\psi(1)]$. Since B has cancellation of full projections by Theorem 2.2.1(c), after conjugation by a unitary in the contractible group $U(M(B \otimes \mathbb{K}))$, we may assume that $\varphi(1) = \psi(1) = p \in \mathcal{P}r(B \otimes \mathbb{K})$. The C^*-algebra $p(B \otimes \mathbb{K})p$ is A-absorbing by [139, Cor.3.1]. It follows that the $*$-homomorphisms $\varphi, \psi : A \to p(B \otimes \mathbb{K})p$ are homotopic by Theorem 2.2.1(b).

(ii) It suffices to prove approximate unitary equivalence for the restrictions of φ and ψ to $A \otimes M_n(\mathbb{C})$ for any $n \geq 1$. Let (e_{ij}) denote the canonical matrix unit of $M_n(\mathbb{C})$, $p_{ij} = \varphi(1 \otimes e_{ij})$, $q_{ij} = \psi(1 \otimes e_{ij})$, $p_n = \varphi(1 \otimes 1_n)$ and $q_n = \psi(1 \otimes 1_n)$. By reasoning as in part (a), we find a partial isometry $v \in B \otimes \mathbb{K}$ such that $v^*v = p_{11}$ and $vv^* = q_{11}$. By [106, Lemma 1.10] there is a partial isometry $w \in M(B \otimes \mathbb{K})$ such that $w^*w = 1 - p_n$ and $ww^* = 1 - q_n$. It follows that $V = w + \sum_{k=1}^{n} q_{k1} v p_{1k}$ is a unitary in $M(B \otimes \mathbb{K})$ such that $V\varphi(1 \otimes x)V^* = \psi(1 \otimes x)$ for all $x \in M_n(\mathbb{C})$. Thus after conjugating φ by a unitary we may assume that $\varphi(1 \otimes x) = \psi(1 \otimes x)$ for all $x \in M_n(\mathbb{C})$. Let us observe that if $a \in A$ and $u \in U(p_{11}(B \otimes \mathbb{K})p_{11})$, then $U = (1 - p_n) + \sum_{k=1}^{n} p_{k1} u p_{1k} \in U(M(B \otimes \mathbb{K}))$ satisfies $U\varphi(a \otimes e_{ij})U^* - \psi(a \otimes e_{ij}) = p_{i1}\big(u\varphi(a \otimes e_{11})u^* - \psi(a \otimes e_{11})\big)p_{1j}$. This reduces our task to proving approximate unitary equivalence for the unital maps $A \otimes e \to p(B \otimes \mathbb{K})p$ induced by φ and ψ, where $p = \varphi(1 \otimes e)$. Since $p(B \otimes \mathbb{K})p$ is A-absorbing, this follows from Theorem 2.2.1(b). □

Next we consider the case when $B = C(X) \otimes A$ in Lemma 2.2.10. We compare two natural multiplicative H-space structures on $\mathrm{End}(A \otimes \mathbb{K})$.

Lemma 2.2.13. *Let X be a topological space, let A be a strongly self-absorbing C^*-algebra and let $\psi\colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K}) \otimes (A \otimes \mathbb{K})$ be a $*$-isomorphism. The two operations $*$ and $\circ$ on $G = [X, \mathrm{End}(A \otimes \mathbb{K})]$ defined by*

$$[\alpha] * [\beta] = [\psi^{-1} \circ (\alpha \otimes \beta) \circ \psi] \quad \textit{and} \quad [\alpha] \circ [\beta] = [\alpha \circ \beta] \ ,$$

where $\alpha \otimes \beta\colon X \to \mathrm{End}((A \otimes \mathbb{K})^{\otimes 2})$ denotes the pointwise tensor product, agree and are both associative and commutative. Moreover, $*$ does not depend on the choice of ψ and is a group operation when restricted to $\mathrm{Aut}(A \otimes \mathbb{K})$.

Proof. First, let $\hat{\psi}$, $\hat{l}$ and $\hat{r}$ be as in the proof of Theorem 2.2.5 and use $\psi = \hat{\psi}(\frac{1}{2})$ in the definition of the operation $*$. Given α, β, δ and $\gamma \in C(X, \mathrm{End}(A \otimes \mathbb{K}))$ we have

$$\begin{aligned}([\alpha] * [\beta]) \circ ([\gamma] * [\delta]) &= [\psi^{-1} \circ (\alpha \otimes \beta) \circ \psi \circ \psi^{-1} \circ (\gamma \otimes \delta) \circ \psi] \\ &= [\psi^{-1} \circ ((\alpha \circ \gamma) \otimes (\beta \circ \delta)) \circ \psi] = ([\alpha] \circ [\gamma]) * ([\beta] \circ [\delta]) \ .\end{aligned}$$

Thus, the Eckmann-Hilton [59] argument will imply that $*$ and $\circ$ agree and are both associative and commutative for this particular choice of ψ if we can show that $\mathrm{id}_{A \otimes \mathbb{K}}$ is a unit for the operation $*$. Just as in the proof of Theorem 2.2.5 we can see that $\hat{\psi}(t/2)^{-1} \circ (\alpha \otimes \mathrm{id}_{A \otimes \mathbb{K}}) \circ \hat{\psi}(t/2)$, $t \in [0, 1]$, is a homotopy from α to $\psi^{-1} \circ (\alpha \otimes \mathrm{id}_{A \otimes \mathbb{K}}) \circ \psi$ with respect to the point-norm topology on $\mathrm{End}(A \otimes \mathbb{K})$ proving that $\mathrm{id}_{A \otimes \mathbb{K}}$ is a right unit. The analogous argument for $\hat{\psi}((t+1)/2)^{-1} \circ (\mathrm{id}_{A \otimes \mathbb{K}} \otimes \alpha) \circ \hat{\psi}((t+1)/2)$ shows that $\mathrm{id}_{A \otimes \mathbb{K}}$ is also a left unit.

If ψ is chosen arbitrarily, we have $\psi = \hat{\psi}(\frac{1}{2}) \circ \kappa$ for some $\kappa \in \mathrm{Aut}(A \otimes \mathbb{K})$. We denote the corresponding operations by $*_{\psi}$ and $*_{\hat{\psi}}$ and have

$$[\alpha] *_{\psi} [\beta] = [\kappa^{-1} \circ \hat{\psi}^{-1}(\tfrac{1}{2}) \circ (\alpha \otimes \beta) \circ \hat{\psi}(\tfrac{1}{2}) \circ \kappa] = [\kappa^{-1}] \circ ([\alpha] *_{\hat{\psi}} [\beta]) \circ [\kappa] = [\alpha] *_{\hat{\psi}} [\beta]$$

by the homotopy commutativity of $\circ$. This proves the independence of $*$ from the choice of the isomorphism ψ. □

We denote by $\approx_u$ the relation of approximate unitary equivalence for $*$-homomorphisms.

Lemma 2.2.14. *Let A be a strongly self-absorbing C^*-algebra. If $p \in A \otimes \mathbb{K}$ is a nonzero projection, the following conditions are equivalent:*

(i) $p(A \otimes \mathbb{K})p \cong A$

(ii) There is $\alpha \in \mathrm{Aut}(A \otimes \mathbb{K})$ *such that* $\alpha(1 \otimes e) = p$.

(iii) $[p] \in K_0(A)_+^\times$

We denote by $\mathcal{P}r(A \otimes \mathbb{K})^\times$ *the set of all projections satisfying these equivalent conditions.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from Corollary 2.2.11.

(ii) $\Rightarrow$ (iii) As an immediate consequence of Lemma 2.2.13 one verifies that the map $\Theta : \pi_0(\mathrm{End}(A \otimes \mathbb{K})) \to K_0(A)$, $\Theta[\alpha] = [\alpha(1 \otimes e)]$ is multiplicative, i.e. $\Theta[\alpha \circ \beta] = \Theta[\alpha]\Theta[\beta]$. Let $q := \alpha^{-1}(1 \otimes e)$. Then $[p][q] = \Theta[\alpha \circ \alpha^{-1}] = \Theta[\mathrm{id}] = [1]$.

(iii) $\Rightarrow$ (i) By assumption there is a full projection $q \in A \otimes K$ such that $[p][q] = [1]$ in $K_0(A)$. By Lemma 2.2.10 there are $\varphi, \psi \in \mathrm{End}(A \otimes \mathbb{K})$ such that $\varphi(1 \otimes e) = p$ and $\psi(1 \otimes e) = q$. Since $[p][q] = [1]$ in $K_0(A)$, it follows that $[\varphi \circ \psi] = [\psi \circ \varphi] = [\mathrm{id}_{\mathrm{A} \otimes \mathbb{K}}] \in [A \otimes \mathbb{K}, A \otimes \mathbb{K}]$. Therefore $\varphi \circ \psi \approx_u \mathrm{id}_{A \otimes \mathbb{K}} \approx_u \psi \circ \varphi$ by Proposition 2.2.12. By [119, Cor.2.3.4] it follows that there is an automorphism $\varphi_0 \in \mathrm{Aut}(A \otimes \mathbb{K})$ such that $\varphi_0 \approx_u \varphi$. Set $p_0 = \varphi_0(1_A \otimes e)$. The map φ_0 induces a $*$-isomorphism $A \cong (1_A \otimes e)(A \otimes \mathbb{K})(1_A \otimes e) \to p_0(A \otimes \mathbb{K})p_0$. We conclude that $A \cong p(A \otimes \mathbb{K})p$ since p_0 is unitarily equivalent to p. $\square$

If A is a separable unital C^*-algebra, Brown, Green and Rieffel [25] showed that the Picard group $\mathrm{Pic}(A)$ is isomorphic to the outer automorphism group of $A \otimes \mathbb{K}$, i.e. $\mathrm{Pic}(A) \cong \mathrm{Out}(A \otimes \mathbb{K}) = \mathrm{Aut}(A \otimes \mathbb{K})/\mathrm{Inn}(A \otimes \mathbb{K})$. One can view $\mathrm{Out}(A)$ as a subgroup of $\mathrm{Pic}(A)$. Kodaka [86] has shown that the coset space $\mathrm{Pic}(A)/\mathrm{Out}(A)$ is in bijection with the Murray-von Neumann equivalence classes of full projections $p \in A \otimes \mathbb{K}$ such that $p(A \otimes \mathbb{K})p \cong A$. From Lemmas 2.2.13 and 2.2.14 we see that if A is strongly self-absorbing, then $\mathrm{Out}(A)$ is a normal subgroup of $\mathrm{Pic}(A)$ and we have:

Corollary 2.2.15. *If A is strongly self-absorbing, then there is an exact sequence of groups*

$$1 \to \mathrm{Out}(A) \to \mathrm{Pic}(A) \to K_0(A)_+^\times \to 1.$$

If moreover A is stably finite, then its normalized trace induces a homomorphism of multiplicative groups from $K_0(A)_+^\times$ onto the fundamental group $\mathcal{F}(A)$ of A defined in [107].

Lemma 2.2.16. *Let A be a strongly self-absorbing C^*-algebra. The sequences* $\mathrm{Aut}_{1 \otimes e}(A \otimes \mathbb{K}) \to \mathrm{Aut}(A \otimes \mathbb{K}) \to \mathcal{P}r(A \otimes \mathbb{K})^\times$ *and* $\mathrm{End}_{1 \otimes e}(A \otimes \mathbb{K}) \to \mathrm{End}(A \otimes \mathbb{K}) \to \mathcal{P}r(A \otimes \mathbb{K})^\times$ *where the first map is the inclusion and the second sends α to $\alpha(1 \otimes e)$ is a locally trivial fiber bundle over a paracompact base space and therefore it is a Hurewicz fibration.*

Proof. Lemma 2.2.14 shows that the map to the base space is surjective. With this remark, the proof is entirely similar the proof of Lemma 2.2.8. □

Corollary 2.2.17. *Let A be a strongly self-absorbing C^*-algebra. Then* $\mathrm{Aut}(A \otimes \mathbb{K}) \simeq \mathrm{End}(A \otimes \mathbb{K})^{\times}$ *has the homotopy type of a CW complex, which is homotopy equivalent to* $K_0(A)^{\times}_{+} \times BU(A)$.

Proof. The equivalence $\mathrm{Aut}(A \otimes \mathbb{K}) \simeq \mathrm{End}(A \otimes \mathbb{K})^{\times}$ follows from Lemma 2.2.16 and Theorem 2.2.5. Moreover, $\mathrm{Aut}(A \otimes \mathbb{K})$ is the coproduct of its path components, all of which are homeomorphic to $\mathrm{Aut}_0(A \otimes \mathbb{K})$. By Theorem 2.2.5 and Lemma 2.2.8, $\mathrm{Aut}_0(A \otimes \mathbb{K})$ is homotopy equivalent to $\mathcal{P}r_0(A \otimes \mathbb{K})$. By Lemma 2.2.14, $\pi_0(\mathcal{P}r(A \otimes \mathbb{K})^{\times}) \cong K_0(A)^{\times}_{+}$. Thus, using Corollary 2.2.9 we have

$$\mathrm{Aut}(A \otimes \mathbb{K}) \simeq \pi_0(\mathcal{P}r(A \otimes \mathbb{K})^{\times}) \times \mathcal{P}r_0(A \otimes \mathbb{K}) \simeq K_0(A)^{\times}_{+} \times BU(A). \qquad \square$$

In the case $A = \mathbb{C}$ this reproves the well-known fact that $\mathrm{Aut}(\mathbb{K}) \simeq BU(1) \simeq K(\mathbb{Z}, 2)$ and hence the only non vanishing homotopy group of $\mathrm{Aut}(\mathbb{K})$ is $\pi_2(\mathrm{Aut}(\mathbb{K})) \cong \pi_2(BU(1)) \cong \pi_1(U(1)) \cong \mathbb{Z}$. At the same time, for $A \neq \mathbb{C}$, we obtain the following.

Theorem 2.2.18. *Let $A \neq \mathbb{C}$ be a strongly self-absorbing C^*-algebra. Then there are isomorphisms of groups*

$$\pi_i(\mathrm{Aut}(A \otimes \mathbb{K})) = \begin{cases} K_0(A)^{\times}_{+} & \text{if } i = 0 \\ K_i(A) & \text{if } i \geq 1 \ . \end{cases}$$

Proof. We have seen in the proof of Corollary 2.2.17 that $\pi_0(\mathrm{Aut}(A \otimes \mathbb{K})) \cong K_0(A)^{\times}_{+}$. If $i \geq 1$, then by Corollary 2.2.17, $\pi_i(\mathrm{Aut}(A \otimes \mathbb{K})) \cong \pi_i(BU(A)) \cong \pi_{i-1}(U(A))$. On the other hand, since A is $\mathcal{Z}$-stable, we have that $\pi_{i-1}(U(A)) \cong K_i(A)$ by [77, Thm.3]. □

Corollary 2.2.19. *Let A be a strongly self-absorbing C^*-algebra. There is an exact sequence of topological groups* $1 \to \mathrm{Aut}_0(A \otimes \mathbb{K}) \to \mathrm{Aut}(A \otimes \mathbb{K}) \to K_0(A)^{\times}_{+} \to 1$.

Remark 2.2.20. The exact sequence $1 \to \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K}) \to \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K}) \to \mathbb{Z}/2 \to 1$ is split, since by [16] there is an order-two automorphism α of $\mathcal{O}_\infty \otimes \mathbb{K}$ such that $\alpha_* = -1$ on $K_0(\mathcal{O}_\infty)$.

Corollary 2.2.21. *Let $A \neq \mathbb{C}$ be a strongly self-absorbing C^*-algebra. The natural map* $\mathrm{Aut}_0(A \otimes \mathbb{K}) \to \mathrm{Aut}_0(\mathcal{O}_\infty \otimes A \otimes \mathbb{K})$ *is a homotopy equivalence.*

Proof. $\mathrm{Aut}_0(A \otimes \mathbb{K}) \to \mathrm{Aut}_0(\mathcal{O}_\infty \otimes A \otimes \mathbb{K})$ is given by $\alpha \mapsto \mathrm{id}_{\mathcal{O}_\infty} \otimes \alpha$. Both spaces have the homotopy type of a CW-complex by Corollary 2.2.9 and they are weakly homotopy equivalent by Thm. 2.2.18. □

Theorem 2.2.22. *Let $A \neq \mathbb{C}$ be a strongly self-absorbing C^*-algebra and let X be a compact metrizable space. The map $\Theta\colon [X, \mathrm{End}(A \otimes \mathbb{K})] \to K_0(C(X) \otimes A)_+$ given by $\Theta([\alpha]) = [\alpha(1 \otimes e)]$ is an isomorphism of commutative semirings. Θ restricts to a group isomorphism $[X, \mathrm{Aut}(A \otimes \mathbb{K})] \to K_0(C(X) \otimes A)_+^\times$. If X is connected, then $K_0(C(X) \otimes A)_+^\times \cong K_0(A)_+^\times \oplus K_0(C_0(X \setminus x_0) \otimes A)$. If A is purely infinite, then $K_0(C(X) \otimes A)_+ = K_0(C(X) \otimes A)$ and Θ is an isomorphism of rings.*

Proof. Let $\psi\colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K})^{\otimes 2}$ and l be as in Lemma 2.2.13. The additivity of Θ is easily verified. Let $\alpha, \beta \in C(X, \mathrm{End}(A \otimes \mathbb{K}))$, then by Lemma 2.2.13:

$$\begin{aligned}[(\alpha \circ \beta)(1 \otimes e)] &= [(\alpha * \beta)(1 \otimes e)] = [\psi^{-1} \circ (\alpha \otimes \beta) \circ \psi(1 \otimes e)] \\ &= [\psi^{-1}(\alpha(1 \otimes e) \otimes \beta(1 \otimes e))] = [\alpha(1 \otimes e)] \cdot [\beta(1 \otimes e)] \ ,\end{aligned}$$

which shows that $\Theta\colon [X, \mathrm{End}(A \otimes \mathbb{K})] \to K_0(C(X) \otimes A)_+$ is a homomorphism of semirings. Let $p \in C(X) \otimes A \otimes \mathbb{K}$ be a full projection. Then, $p(C(X) \otimes A \otimes \mathbb{K})p$ is A-absorbing by [139, Cor.3.1]. It follows that Θ is surjective by Lemma 2.2.10. For injectivity we apply Proposition 2.2.12(i).

Next we show that the image of the restriction of Θ to $[X, \mathrm{Aut}(A \otimes \mathbb{K})]$ coincides with $K_0(C(X) \otimes \mathbb{K})_+^\times$. Let $p \in \mathcal{P}r(C(X) \otimes A \otimes \mathbb{K})^\times$. By assumption, there is $q \in \mathcal{P}r(C(X) \otimes A \otimes \mathbb{K})^\times$ such that $[p][q] = 1$ in the ring $K_0(C(X) \otimes A)$. By Lemma 2.2.10 there are $\varphi, \psi \in \mathrm{Hom}(A \otimes \mathbb{K}, C(X) \otimes A \otimes \mathbb{K})$ such that $\varphi(1 \otimes e) = p$ and $\psi(1 \otimes e) = q$. Let $\tilde{\varphi}, \tilde{\psi} \in \mathrm{End}_{C(X)}(C(X) \otimes A \otimes \mathbb{K})$ be the unique $C(X)$-linear extensions of φ and ψ. Note that if $\iota : A \otimes \mathbb{K} \to C(X) \otimes A \otimes \mathbb{K}$ is the inclusion $\iota(a) = 1_{C(X)} \otimes a$, then $\tilde{\iota} = \mathrm{id}_{C(X) \otimes A \otimes \mathbb{K}}$. Since $[p][q] = [1]$ in $K_0(C(X) \otimes A)$, it follows that $[\tilde{\varphi} \circ \psi] = [\tilde{\psi} \circ \varphi] = [\iota] \in [A \otimes \mathbb{K}, C(X) \otimes A \otimes \mathbb{K}]$. By Proposition 2.2.12(ii) it follows that $\tilde{\varphi} \circ \psi \approx_u \iota \approx_u \tilde{\psi} \circ \varphi$. This clearly implies that $\tilde{\varphi} \circ \tilde{\psi} \approx_u \mathrm{id}_{C(X) \otimes A \otimes \mathbb{K}} \approx_u \tilde{\psi} \circ \tilde{\varphi}$. By [119, Cor.2.3.4] it follows that there is an automorphism $\alpha \in \mathrm{Aut}_{C(X)}(C(X) \otimes A \otimes \mathbb{K})$ such that $\alpha \approx_u \tilde{\varphi}$. In particular we have that $[\alpha(1 \otimes e)] = [\tilde{\varphi}(1 \otimes e)] = [p]$.

It remains to verify the isomorphism $K_0(C(X) \otimes A)_+^\times \cong K_0(A)_+^\times \oplus K_0(C_0(X \setminus x_0) \otimes A)$. Evaluation at x_0 induces a split exact sequence $0 \to K_0(C_0(X \setminus x_0) \otimes A) \to K_0(C(X) \otimes A) \to K_0(A) \to 0$. Arguing as in the proof of [42, Prop.5.6], one verifies that $K_0(C_0(X \setminus x_0) \otimes A)$ is a nil-ideal of the ring $K_0(C(X) \otimes A)$. Thus an element $\sigma \in K_0(C(X) \otimes A)$ is invertible if and only if its restriction $\sigma_{x_0} \in K_0(A)$ is invertible. Consequently $K_0(C(X) \otimes A)^\times \cong K_0(A)^\times \oplus K_0(C_0(X \setminus x_0) \otimes A)$. It remains to verify that an element $\sigma \in K_0(C(X) \otimes A)$ is positive if $\sigma_{x_0} \in K_0(A)_+ \setminus \{0\}$. It suffices to consider the case when A is stably finite. Let τ denote the unique trace state of A. Its extension to a trace state on $A \otimes M_n(\mathbb{C})$ is denoted again by τ. Then any continuous trace η on $C(X) \otimes A \otimes M_n(\mathbb{C})$ is of the form $\eta(f) = \int_X \tau(f) d\mu$ for some finite Borel measure μ on X. Write $\sigma = [p] - [q]$ where $p, q \in \mathcal{P}r(C(X) \otimes A \otimes M_n(\mathbb{C}))$ are full projections. Let r be a nonzero projection in $A \otimes M_n(\mathbb{C})$ such that $[p(x_0)] - [q(x_0)] = [r]$. Since X is connected it follows that $[p(x)] - [q(x)] = [r] \in K_0(A)$ for all $x \in X$. From this we see that any point $x \in X$ has a closed neighborhood V such that $[p_V] - [q_V] = [r] \in K_0(C(V) \otimes A)$. Since $\tau(r) > 0$ it follows immediately that $\eta(p) > \eta(q)$ for all nonzero finite traces η on $A \otimes M_n(\mathbb{C})$. We apply Corollaries 4.9 and 4.10 of [120] to conclude that $[p] - [q] \in K_0(C(X) \otimes A)_+$. $\square$

Corollary 2.2.23. *Let X be a compact connected metrizable space. Then there are isomorphism of multiplicative groups*

$$[X, \mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})] \cong K^0(X)^\times_+ = 1 + \widetilde{K}^0(X),$$

$$[X, \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})] \cong K^0(X)^\times = \pm 1 + \widetilde{K}^0(X).$$

2.2.5 The topological group $\mathrm{Aut}(A \otimes \mathbb{K})$ is well-pointed

Since we would like to apply the nerve construction to obtain classifying spaces of the topological monoids $\mathrm{Aut}(A \otimes \mathbb{K})$ and $\mathrm{End}(A \otimes \mathbb{K})^\times$, we will need to show that $\mathrm{Aut}(A \otimes \mathbb{K})$ is well-pointed. This notion is defined as follows:

Definition 2.2.24. Let X be a topological space, $A \subset X$ a closed subspace. The pair (X, A) is called a *neighborhood deformation retract* (or *NDR-pair* for short) if there is a map $u\colon X \to I = [0,1]$ such that $u^{-1}(0) = A$ and a homotopy $H\colon X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(a, t) = a$ for $a \in A$ and $t \in I$ and $H(x, 1) \in A$ if $u(x) < 1$. A pointed topological space X with basepoint $x_0 \in X$ is said to have a *non-degenerate basepoint* or to be *well-pointed* if the pair (X, x_0) is an NDR-pair.

Recall that a neighborhood V of x_0 deformation retracts to x_0 if there is a continuous map $h : V \times I \to V$ such that $h(x, 0) = x$, $h(x_0, t) = x_0$ and $h(x, 1) = x_0$ for all $x \in V$ and $t \in I$. The following Lemma is contained in [131, Thm.2].

Lemma 2.2.25. *Let (X, x_0) be a pointed topological space together with a continuous map $v\colon X \to I$ such that $x_0 = v^{-1}(0)$ and $V = \{x \in X\colon v(x) < 1\}$ deformation retracts to x_0. Then (X, x_0) is an NDR-pair.*

Proposition 2.2.26. *Let A be a strongly self-absorbing C^*-algebra. Then the topological monoids $\mathrm{Aut}(A \otimes \mathbb{K})$ and $\mathrm{End}(A \otimes \mathbb{K})^\times$ are well-pointed.*

Proof. We will prove this for $\mathrm{Aut}(A \otimes \mathbb{K})$, but the proof for $\mathrm{End}(A \otimes \mathbb{K})^\times$ is entirely similar. Let $e \in \mathbb{K}$ be a rank-1 projection and set $p_0 = 1 \otimes e$. Let $U = \{p \in \mathcal{P}r(A \otimes \mathbb{K}) \mid \|p - p_0\| < 1/2\}$. If $\pi\colon \mathrm{Aut}_0(A \otimes \mathbb{K}) \to \mathcal{P}r_0(A \otimes \mathbb{K})$ denotes the map $\beta \mapsto \beta(p_0)$, we will show that $\pi^{-1}(U)$ deformation retracts to $\mathrm{id}_{A \otimes \mathbb{K}} \in \mathrm{Aut}_0(A \otimes \mathbb{K})$. As we have seen in the proof of

Lemma 2.2.8, the principal bundle $\mathrm{Aut}_0(A \otimes \mathbb{K}) \to \mathcal{P}r_0(A \otimes \mathbb{K})$ trivializes over U, i.e. there exists a homeomorphism $\pi^{-1}(U) \to U \times \mathrm{Aut}_{p_0}(A \otimes \mathbb{K})$ sending $\mathrm{id}_{A \otimes \mathbb{K}}$ to $(p_0, \mathrm{id}_{A \otimes \mathbb{K}})$. Thus, it suffices to show that the right hand side retracts. Let χ be the characteristic function of $(1/2, 1]$. Then $h(p,t) = \chi((1-t)p + tp_0)$ is a deformation retraction of U into p_0. This is well-defined since $1/2$ is not in the spectrum of $a = (1-t)p + tp_0$ as seen from the estimate $\|(1-2a) - (1-2p_0)\| < 1$. We have shown in Theorem 2.2.5 that $\mathrm{Aut}_{p_0}(A \otimes \mathbb{K})$ deformation retracts to $\mathrm{id}_{A \otimes \mathbb{K}}$. Combining these homotopies we end up with a deformation retraction of $\pi^{-1}(U)$ into $\mathrm{id}_{A \otimes \mathbb{K}}$. Let d be a metric for $\mathrm{Aut}(A \otimes \mathbb{K})$. Then $v\colon \mathrm{Aut}(A \otimes \mathbb{K}) \to [0,1]$, $v(\alpha) = \max\{\min\{d(\alpha, \mathrm{id}_{A \otimes \mathbb{K}}), 1/2\}, \min\{1, 2\|\alpha(p_0) - p_0\|\}\}$ and $V := \pi^{-1}(U)$ satisfy the conditions of Lemma 2.2.25 relative to the basepoint $\mathrm{id}_{A \otimes \mathbb{K}}$. □

2.3 The infinite loop space structure of $B\mathrm{Aut}(A \otimes \mathbb{K})$

2.3.1 Permutative categories and infinite loop spaces

We will show that $B\mathrm{Aut}(A \otimes \mathbb{K})$ is an infinite loop space in the sense of the following definition [2].

Definition 2.3.1. A topological space $E = E_0$ is called an *infinite loop space*, if there exists a sequence of spaces E_i, $i \in \mathbb{N}$, such that $E_i \simeq \Omega E_{i+1}$ for all $i \in \mathbb{N}_0$ ($\simeq$ denotes homotopy equivalence).

The importance of these spaces lies in the fact, that they represent generalized cohomology theories, i.e. for a CW-complex X, the homotopy classes of maps $E^i(X) := [X, E_i]$ are abelian groups and the functor $X \mapsto E^\bullet(X)$ is a cohomology theory. There may be many inequivalent delooping sequences starting with the same E_0 leading to different theories. The sequence of spaces E_i forms a *connective Ω-spectrum.* There is a well-developed theory to detect whether a space belongs to this class [99, 129]. One of the main sources for infinite loop spaces are classifying spaces of topological strict symmetric monoidal categories, called *permutative categories* in [101].

A *topological category* has a space of objects, a space of morphisms and continuous source, target and identity maps. Such a category $\mathcal{C}$ carries a *strict monoidal structure* if it comes equipped with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that satisfies the analogues of associativity and unitality known for monoids. The strictness refers to the fact that these hold on the nose, not only up to natural transformations. $\mathcal{C}$ is called *symmetric* if it comes equipped with a natural transformation $c\colon \otimes \circ \tau \to \otimes$, where $\tau\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is switching the factors. This should behave like a permutation on n-fold tensor products. We will assume all permutative categories to be *well-pointed* in the sense that the map $\mathrm{obj}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$, $x \to \mathrm{id}_x$ is a cofibration. For a precise definition, we refer the reader to [101, Def.1]. Note further that all categories we consider in this paper will be *small*.

Any topological category $\mathcal{C}$ can be turned into a simplicial space $N_\bullet\mathcal{C}$ via the nerve construction. Let $N_0\,\mathcal{C} = \mathrm{obj}(\mathcal{C})$, $N_1\mathcal{C} = \mathrm{mor}(\mathcal{C})$ and

$$N_k\,\mathcal{C} = \{(f_1,\dots,f_k) \in \mathrm{mor}(\mathcal{C}) \times \dots \times \mathrm{mor}(\mathcal{C}) \mid s(f_i) = t(f_{i+1})\}\ .$$

The face maps $d_i^k \colon N_k\,\mathcal{C} \to N_{k-1}\,\mathcal{C}$ and degeneracies $s_i^k \colon N_k\,\mathcal{C} \to N_{k+1}\,\mathcal{C}$ are induced by composition of successive maps and insertion of identities respectively. The geometric realization of a simplicial space $X_\bullet$ is defined by

$$|X_\bullet| = \left(\coprod_{k=0} X_k \times \Delta_k\right) / \sim$$

where $\Delta_k \subset \mathbb{R}^{k+1}$ denotes the standard k-simplex and the equivalence relation is generated by $(d_i^k x, u) \sim (x, \partial_i^{k-1} u)$ and $(s_i^l y, v) \sim (y, \sigma_i^{l+1} v)$ for $x \in X_k$, $u \in \Delta_{k-1}$, $y \in X_l$, $v \in X_{l+1}$, where δ_i and σ_i are the coface and codegeneracy maps on the standard simplex. For details about this construction we refer the reader to [99, Sec.11].

The space $|N_\bullet\mathcal{C}|$ associated to a category $\mathcal{C}$ is called the *classifying space of* $\mathcal{C}$. If $\mathcal{C}$ is the category associated to a monoid M, then we denote $|N_\bullet\mathcal{C}|$ by BM. Having a monoidal structure on $\mathcal{C}$ yields the following.

Lemma 2.3.2. *Let $\mathcal{C}$ be a strict monoidal topological category. Then $|N_\bullet\mathcal{C}|$ is a topological monoid.*

Proof. The nerve construction $N_\bullet$ preserves products in the sense that the projection functors $\pi_i \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induce a levelwise homeomorphism $N_\bullet(\mathcal{C} \times \mathcal{C}) \to N_\bullet\mathcal{C} \times N_\bullet\mathcal{C}$. Therefore $N_\bullet\mathcal{C}$ is a simplicial topological monoid and the lemma follows from [99, Cor.11.7]. □

A permutative category $\mathcal{C}$ provides an input for infinite loop space machines [101, Def.2]. Due to the above Lemma, there is a classifying space $B|N_\bullet\mathcal{C}|$. The following has been proven by Segal [129] and May [100, Thm.4.10]

Theorem 2.3.3. *Let $\mathcal{C}$ be a permutative category. Then $\Omega B|N_\bullet\mathcal{C}|$ is an infinite loop space. Moreover, if $\pi_0(|N_\bullet\mathcal{C}|)$ is a group, then the map $|N_\bullet\mathcal{C}| \to \Omega B|N_\bullet\mathcal{C}|$ induced by the inclusion of the 1-skeleton $S^1 \times |N_\bullet\mathcal{C}| \to B|N_\bullet\mathcal{C}|$ is a homotopy equivalence of H-spaces.*

2.3.2 The tensor product of $A \otimes \mathbb{K}$-bundles

Let A be a strongly self-absorbing C^*-algebra, X be a topological space and let P_1 and P_2 be principal $\mathrm{Aut}(A \otimes \mathbb{K})$-bundles over X. Fix an isomorphism $\psi \colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K}) \otimes (A \otimes \mathbb{K})$.

This choice induces a tensor product operation on principal $\mathrm{Aut}(A \otimes \mathbb{K})$-bundles in the following way. Note that $P_1 \times_X P_2 \to X$ is a principal $\mathrm{Aut}(A \otimes \mathbb{K}) \times \mathrm{Aut}(A \otimes \mathbb{K})$-bundle and ψ induces a group homomorphism

$$\mathrm{Ad}_{\psi^{-1}} \colon \mathrm{Aut}(A \otimes \mathbb{K}) \times \mathrm{Aut}(A \otimes \mathbb{K}) \to \mathrm{Aut}(A \otimes \mathbb{K}) \quad ; \quad (\alpha, \beta) \mapsto \psi^{-1} \circ (\alpha \otimes \beta) \circ \psi \ .$$

Now let

$$P_1 \otimes_\psi P_2 := (P_1 \times_X P_2) \times_{\mathrm{Ad}_{\psi^{-1}}} \mathrm{Aut}(A \otimes \mathbb{K}) = ((P_1 \times_X P_2) \times \mathrm{Aut}(A \otimes \mathbb{K}))/ \sim$$

where the equivalence relation is $(p_1\, \alpha, p_2\, \beta, \gamma) \sim (p_1, p_2, \mathrm{Ad}_{\psi^{-1}}(\alpha, \beta)\, \gamma)$ for all $(p_1, p_2) \in P_1 \times_X P_2$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(A \otimes \mathbb{K})$. This is a delooped version of the operation $*$ from Lemma 2.2.13.

Due to the choice of ψ, which was arbitrary, $\otimes_\psi$ can not be associative. We will show, however, that – just like $*$ – it is *homotopy associative* and also *homotopy unital*.

To obtain a model for the classifying space $B\mathrm{Aut}(A \otimes \mathbb{K})$, let $\mathcal{B}$ be the topological category, which has as its object space just a single point and the group $\mathrm{Aut}(A \otimes \mathbb{K})$ as its morphism space. Since we have shown that $\mathrm{Aut}(A \otimes \mathbb{K})$ is well-pointed (see Proposition 2.2.26), [103, Prop.7.5 and Thm.8.2] implies that the geometric realization $|N_\bullet\mathcal{B}|$ has in fact the homotopy type of a classifying space for principal $\mathrm{Aut}(A \otimes \mathbb{K})$-bundles, i.e.

$$B\mathrm{Aut}(A \otimes \mathbb{K}) = |N\mathcal{B}_\bullet| \ .$$

Choosing an isomorphism $\psi \colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K}) \otimes (A \otimes \mathbb{K})$, we can define a functor $\otimes_\psi \colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ just as above, which acts on morphisms $\alpha, \beta \in \mathrm{Aut}(A \otimes \mathbb{K})$ by

$$(\alpha, \beta) \mapsto \alpha \otimes_\psi \beta := \mathrm{Ad}_{\psi^{-1}}(\alpha, \beta) \ .$$

This is in fact functorial since composition is well-behaved with respect to the tensor product in the following way

$$\begin{aligned} (\alpha \circ \alpha') \otimes_\psi (\beta \circ \beta') &= \psi^{-1} \circ (\alpha \circ \alpha') \otimes (\beta \circ \beta') \circ \psi \\ = \psi^{-1} \circ (\alpha \otimes \beta) \circ \psi \circ \psi^{-1} &\circ (\alpha' \otimes \beta') \circ \psi = (\alpha \otimes_\psi \beta) \circ (\alpha' \otimes_\psi \beta') \ . \end{aligned}$$

The functor induces a multiplication map on the geometric realization

$$\mu_\psi \colon B\mathrm{Aut}(A \otimes \mathbb{K}) \times B\mathrm{Aut}(A \otimes \mathbb{K}) \to B\mathrm{Aut}(A \otimes \mathbb{K}) \ .$$

Observe that a path connecting two isomorphisms $\psi, \psi' \in \mathrm{Iso}(\mathrm{Aut}(A \otimes \mathbb{K}), \mathrm{Aut}(A \otimes \mathbb{K})^{\otimes 2})$ induces a homotopy of functors $\mathcal{B} \times \mathcal{B} \times I \to \mathcal{B}$, where I here is the category, which has

$[0,1]$ as its object space and only identities as morphisms. After geometric realization this in turn yields a homotopy between μ_ψ and $\mu_{\psi'}$ (observe that $|I| \cong [0,1]$).

Lemma 2.3.4. *Let A be a strongly self-absorbing C^*-algebra and let $\mathcal{B}$ be the category defined above. Let $\psi\colon A\otimes\mathbb{K} \to (A\otimes\mathbb{K})\otimes(A\otimes\mathbb{K})$ be an isomorphism, then μ_ψ defines an H-space structure on $B\mathrm{Aut}(A\otimes\mathbb{K})$, which has the basepoint of $B\mathrm{Aut}(A\otimes\mathbb{K})$ as a homotopy unit and agrees with the H-space structure induced by the tensor product $\otimes_\psi$ of $A\otimes\mathbb{K}$-bundles. Different choices of ψ yield homotopy equivalent H-space structures.*

Proof. The proof of this statement is very similar to the one of Lemma 2.2.13, but we have to take care that the homotopies we use run through *functors* on $\mathcal{B}$. Let $\hat{\psi}$, $\hat{l}$ and $\hat{r}$ be just as in Theorem 2.2.5 and consider $\psi = \hat{\psi}(\frac{1}{2})$ first. By Theorem 2.2.5, there is a path between $(\psi\otimes\mathrm{id}_{A\otimes\mathbb{K}})\circ\psi$ and $(\mathrm{id}_{A\otimes\mathbb{K}}\otimes\psi)\circ\psi$, since both these morphisms map $1\otimes e$ to $(1\otimes e)^{\otimes 3}$. This proves the homotopy associativity in this case.

To prove that the basepoint provides a homotopy unit we have to show that the two functors $\alpha\mapsto\alpha\otimes_\psi\mathrm{id}_{A\otimes\mathbb{K}}$ and $\alpha\mapsto\mathrm{id}_{A\otimes\mathbb{K}}\otimes_\psi\alpha$ are both homotopic to the identity functor. The argument for this is the same as in the proof of Lemma 2.2.13.

Now let $\psi\colon A\otimes\mathbb{K}\to(A\otimes\mathbb{K})\otimes(A\otimes\mathbb{K})$ be an arbitrary isomorphism. As in Lemma 2.2.13 we have $\psi = \hat{\psi}\circ\kappa$ for some automorphism $\kappa\in\mathrm{Aut}(A\otimes\mathbb{K})$. If we denote homotopic functors by $\sim$, we have

$$\begin{aligned}\alpha\otimes_\psi\beta &= \kappa^{-1}\circ(\alpha\otimes_{\hat{\psi}}\beta)\circ\kappa \sim (\kappa\otimes_{\hat{\psi}}\mathrm{id}_{A\otimes\mathbb{K}})^{-1}\circ(\mathrm{id}_{A\otimes\mathbb{K}}\otimes_{\hat{\psi}}(\alpha\otimes_{\hat{\psi}}\beta))\circ(\kappa\otimes_{\hat{\psi}}\mathrm{id}_{A\otimes\mathbb{K}})\\ &= \mathrm{id}_{A\otimes\mathbb{K}}\otimes_{\hat{\psi}}(\alpha\otimes_{\hat{\psi}}\beta)\sim(\alpha\otimes_{\hat{\psi}}\beta)\ .\end{aligned}$$

Note that every stage of this homotopy provides functors $\mathcal{B}\times\mathcal{B}\to\mathcal{B}$. Geometrically realizing this homotopy we see that different choices of ψ yield the same H-space structure up to homotopy.

Let $EG\to BG$ be the universal G-bundle where $G = \mathrm{Aut}(A\otimes\mathbb{K})$ [103, section 7]. Using its simplicial description, we see that $\mu_\psi^*EG\cong\pi_1^*EG\otimes_\psi\pi_2^*EG$, where $\pi_i\colon BG\times BG\to BG$ are the canonical projections. Now, given two classifying maps $f_k\colon X\to BG$ and the diagonal map $\Delta\colon BG\to BG\times BG$, we have

$$(\mu_\psi\circ(f_1,f_2)\circ\Delta)^*EG = \Delta^*\circ(f_1^*,f_2^*)\circ\mu_\psi^*EG = f_1^*EG\otimes_\psi f_2^*EG$$

proving that the multiplication induced by the H-space structure on $[X, B\mathrm{Aut}(A\otimes\mathbb{K})]$ agrees with the tensor product $\otimes_\psi$. $\square$

Definition 2.3.5. For a strongly self-absorbing C^*-algebra A we define $(\mathcal{B}un_X(A\otimes\mathbb{K}),\otimes)$ to be the monoid of isomorphism classes of principal $\mathrm{Aut}(A\otimes\mathbb{K})$-bundles with respect to the tensor product induced by $\otimes_\psi$. By the above Lemma, this is independent of the choice of ψ.

To apply the infinite loop space machine, we need a permutative category encoding the operation $\otimes_\psi$. Let $\mathcal{B}_\otimes$ be the category, which has $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ as its object space (where $n \in \mathbb{N}_0$ should be thought of as $(A \otimes \mathbb{K})^{\otimes n}$ with $(A \otimes \mathbb{K})^{\otimes 0} = \mathbb{C}$). The morphisms from n to m are given by $\hom_{\mathcal{B}_\otimes}(n, m) = \{\alpha \in \mathrm{Hom}((A \otimes \mathbb{K})^{\otimes n}, (A \otimes \mathbb{K})^{\otimes m}) \mid KK(\alpha) \text{ invertible}\}$ for $n \geq 1$ and $\hom_{\mathcal{B}_\otimes}(0, m) = \{\alpha \in \mathrm{Hom}(\mathbb{C}, (A \otimes \mathbb{K})^{\otimes m}) \mid [\alpha(1)] \in K_0((A \otimes \mathbb{K})^{\otimes m})^\times\}$ for $n = 0$. We equip these spaces with the point-norm topology. The ordinary tensor product of $*$-homomorphisms induces a *strict* monoidal structure $\otimes \colon \mathcal{B}_\otimes \times \mathcal{B}_\otimes \to \mathcal{B}_\otimes$, where $n \otimes m = n + m$. Likewise, we have a symmetry $c_{n,m}$ on $\mathcal{B}_\otimes$, where $c_{n,m} \in \mathrm{Aut}((A \otimes \mathbb{K})^{\otimes(n+m)})$ is the automorphism $(A \otimes \mathbb{K})^{\otimes n} \otimes (A \otimes \mathbb{K})^{\otimes m} \to (A \otimes \mathbb{K})^{\otimes m} \otimes (A \otimes \mathbb{K})^{\otimes n}$ switching the two factors. With these definitions $\mathcal{B}_\otimes$ becomes a permutative category. Define

$$B\mathrm{End}(A \otimes \mathbb{K})^\times_\otimes := |N_\bullet \mathcal{B}_\otimes| \ .$$

Lemma 2.3.6. *The inclusion functor $J : \mathcal{B} \to \mathcal{B}_\otimes$ induces a homotopy equivalence of the corresponding classifying spaces $B\mathrm{Aut}(A \otimes \mathbb{K}) \to B\mathrm{End}(A \otimes \mathbb{K})^\times_\otimes$. Given an isomorphism $\psi \colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K})^{\otimes 2}$, the diagram*

$$\begin{tikzcd} \mathcal{B} \times \mathcal{B} \arrow[r, "\otimes_\psi"] \arrow[d, "J \times J"'] & \mathcal{B} \arrow[d, "J"] \\ \mathcal{B}_\otimes \times \mathcal{B}_\otimes \arrow[r, "\otimes"] & \mathcal{B}_\otimes \end{tikzcd} \tag{2.2}$$

commutes up to a natural transformation. In particular, the H-space structure of $B\mathrm{Aut}(A \otimes \mathbb{K})$ agrees with the one on $B\mathrm{End}(A \otimes \mathbb{K})^\times_\otimes$ up to homotopy.

Proof. To prove the first statement we will construct auxiliary categories $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{B}^1_\otimes$ together with inclusion functors $\mathcal{B} \to \mathcal{E} \to \mathcal{H} \to \mathcal{B}^1_\otimes \to \mathcal{B}_\otimes$ that give a factorization of J. We then show that each of these functors induces a homotopy equivalence on classifying spaces. We will use the following two facts.

(a) Given two topological categories $\mathcal{C}$ and $\mathcal{D}$ together with continuous functors $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{C}$ and natural transformations $F \circ G \Rightarrow \mathrm{id}_\mathcal{D}$, $G \circ F \Rightarrow \mathrm{id}_\mathcal{C}$, it follows that F and G induce a homotopy equivalence of the corresponding classifying spaces. This is a corollary of [128, Prop.2.1].

(b) Consider two good simplicial spaces $X_\bullet$ and $Y_\bullet$ ("good" refers to [129, Definition A.4]) together with a simplicial map $f_\bullet \colon X_\bullet \to Y_\bullet$. If $f_n \colon X_n \to Y_n$ is a homotopy equivalence for each $n \in \mathbb{N}_0$, then $|f_\bullet| \colon |X_\bullet| \to |Y_\bullet|$ is also a homotopy equivalence. This is proven in [129, Proposition A.1 (ii) and (iv)]. Note in particular, that the nerve $N_\bullet \mathcal{C}$ of a topological category $\mathcal{C}$ is good, if the map $\mathrm{obj}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$, which sends an object to the identity on it, is a cofibration. This holds for all categories in this proof by Proposition 2.2.26.

The object space of $\mathcal{E}$ consists of a single point and its morphism space is $\mathrm{End}(A\otimes\mathbb{K})^\times$. From Lemma 2.2.16 and Theorem 2.2.5 we obtain that $\mathrm{Aut}(A\otimes\mathbb{K}) \to \mathrm{End}(A\otimes\mathbb{K})^\times$ is a homotopy equivalence. Thus each component $N_k\mathcal{B} \to N_k\mathcal{E}$ of the simplicial map $N_\bullet\mathcal{B} \to N_\bullet\mathcal{E}$ induced by the inclusion functor $\mathcal{B} \to \mathcal{E}$ is a homotopy equivalence of spaces. This yields a homotopy equivalence $|N_\bullet\mathcal{B}| \to |N_\bullet\mathcal{E}|$ by (b) above.

The category $\mathcal{B}^1_\otimes$ is the full subcategory of $\mathcal{B}_\otimes$ containing the objects 0 and 1. To see that the inclusion functor $\iota\colon \mathcal{B}^1_\otimes \to \mathcal{B}_\otimes$ is an equivalence of categories, we argue as follows: Define an inverse functor $\tau\colon \mathcal{B}_\otimes \to \mathcal{B}^1_\otimes$ for ι, such that $\tau(m) = \min\{m,1\}$ on objects. Let $\psi_0 = \mathrm{id}_\mathbb{C}$ and fix isomorphisms $\psi_k\colon A\otimes\mathbb{K} \to (A\otimes\mathbb{K})^{\otimes k}$ for $k\in\mathbb{N}$ with $\psi_1 = \mathrm{id}_{A\otimes\mathbb{K}}$. Define $\tau(\beta) = \psi_\ell^{-1}\circ\beta\circ\psi_k$ for $\beta \in \hom_{\mathcal{B}_\otimes}(k,\ell)$. We have $\tau\circ\iota = \mathrm{id}_{\mathcal{B}^1_\otimes}$ and the ψ_k yield a natural transformation $\iota\circ\tau \Rightarrow \mathrm{id}_{\mathcal{B}_\otimes}$. Thus, the map $|N_\bullet\mathcal{B}^1_\otimes| \to |N_\bullet\mathcal{B}_\otimes|$ induced by ι is a homotopy equivalence by (a).

Let $\mathcal{H}$ be the topological category with object space $\{0,1\}$ and morphism spaces

$$\hom_\mathcal{H}(0,0) = \{\mathrm{id}_{A\otimes\mathbb{K}}\} \quad , \quad \hom_\mathcal{H}(0,1) = \hom_\mathcal{H}(1,1) = \mathrm{End}(A\otimes\mathbb{K})^\times$$

and $\hom(1,0) = \emptyset$. The composition is induced by the composition in $\mathrm{End}(A\otimes\mathbb{K})^\times$. Note that there is a restriction functor $\mathcal{H} \to \mathcal{B}^1_\otimes$, which takes $\beta \in \hom_\mathcal{H}(0,1) = \mathrm{End}(A\otimes\mathbb{K})^\times$ to $\widetilde{\beta} \in \hom_{\mathcal{B}^1_\otimes}(0,1) = \hom_{\mathcal{B}_\otimes}(0,1)$, where $\widetilde{\beta}(\lambda) = \lambda\beta(1\otimes e)$ for $\lambda\in\mathbb{C}$. It maps the spaces $\hom_\mathcal{H}(0,0)$ and $\hom_\mathcal{H}(1,1)$ identically onto $\hom_{\mathcal{B}^1_\otimes}(0,0)$ and $\hom_{\mathcal{B}^1_\otimes}(1,1)$ respectively. By Lemma 2.2.16 and Theorem 2.2.5 the restriction map $\mathrm{End}(A\otimes\mathbb{K})^\times \to \mathrm{Hom}(\mathbb{C}, A\otimes\mathbb{K})^\times \cong \mathcal{P}r(A\otimes\mathbb{K})^\times$ is a homotopy equivalence. Therefore the simplicial map $N_k\mathcal{H} \to N_k\mathcal{B}^1_\otimes$ is a homotopy equivalence for each k, and hence $|N_\bullet\mathcal{H}| \to |N_\bullet\mathcal{B}^1_\otimes|$ is a homotopy equivalence by (b).

Let $\iota_\mathcal{E}\colon \mathcal{E} \to \mathcal{H}$ be the inclusion functor. Let $\tau_\mathcal{E}\colon \mathcal{H} \to \mathcal{E}$ be the functor, which maps the two objects of $\mathcal{H}$ to the one of $\mathcal{E}$ and which embeds the spaces $\hom_\mathcal{H}(0,0)$, $\hom_\mathcal{H}(0,1)$ and $\hom_\mathcal{H}(1,1)$ into $\mathrm{End}(A\otimes\mathbb{K})^\times$ in a canonical way. We have $\tau_\mathcal{E}\circ\iota_\mathcal{E} = \mathrm{id}_\mathcal{E}$. There is a natural transformation $\kappa\colon \mathrm{id}_\mathcal{H} \Rightarrow \iota_\mathcal{E}\circ\tau_\mathcal{E}$ with $\kappa_1 = \mathrm{id}_{A\otimes\mathbb{K}} \in \hom_\mathcal{H}(1,1)$ and $\kappa_0 = \mathrm{id}_{A\otimes\mathbb{K}} \in \hom_\mathcal{H}(0,1)$. It follows that $\iota_\mathcal{E}$ also induces an equivalence on classifying spaces by (a). This concludes the proof of the first statement.

Let β_1, β_2 be morphisms in $\mathcal{B}$, then $(J\circ\otimes_\psi)(\beta_1,\beta_2) = \beta_1\otimes_\psi\beta_2 = \psi^{-1}\circ(\beta_1\otimes\beta_2)\circ\psi \in \hom_{\mathcal{B}_\otimes}(1,1)$, whereas $\otimes\circ(J\times J)(\beta_1,\beta_2) = \beta_1\otimes\beta_2 \in \hom_{\mathcal{B}_\otimes}(2,2)$ and $\psi \in \hom_{\mathcal{B}_\otimes}(1,2)$ provides a natural transformation $J\circ\otimes_\psi \Rightarrow \otimes\circ(J\times J)$. Thus these two functors induce homotopic maps of classifying spaces by [128, Prop.2.1]. This completes the proof. □

Corollary 2.3.7. *The space $B\mathrm{Aut}(A\otimes\mathbb{K})$ inherits an infinite loop space structure via the homotopy equivalence $B\mathrm{Aut}(A\otimes\mathbb{K}) \to B\mathrm{End}(A\otimes\mathbb{K})^\times_\otimes$ in such a way that the induced*

H-space structure of $B\mathrm{Aut}(A\otimes\mathbb{K})$ agrees with the one given by μ_ψ.

Proof. By Theorem 2.3.3, $B\mathrm{End}(A\otimes\mathbb{K})^\times_\otimes$ is an infinite loop space with H-space structure induced by the tensor product of $\mathcal{B}_\otimes$. By Lemma 2.3.6, $B\mathrm{Aut}(A\otimes\mathbb{K})\to B\mathrm{End}(A\otimes\mathbb{K})^\times_\otimes$ is a homotopy equivalence and a map of H-spaces. □

Theorem 2.3.8. *Let A be a strongly self-absorbing C^*-algebra.*

(a) The monoid $(\mathcal{B}un_X(A\otimes\mathbb{K}),\otimes)$ of isomorphism classes of principal $\mathrm{Aut}(A\otimes\mathbb{K})$-bundles is an abelian group.

(b) $B\mathrm{Aut}(A\otimes\mathbb{K})$ is the first space in a spectrum defining a cohomology theory $E_A^\bullet$ with $E_A^0(X)=[X,\mathrm{Aut}(A\otimes\mathbb{K})]$ and $E_A^1(X)=\mathcal{B}un_X(A\otimes\mathbb{K})$.

(c) If X is a compact metrizable space and $A\neq\mathbb{C}$, then $E_A^0(X)\cong K_0(C(X)\otimes A)^\times_+$.

Proof. By Corollary 2.3.7 the space $B\mathrm{Aut}(A\otimes\mathbb{K})$ is an infinite loop space with H-space structure given by with $\otimes_\psi$, which implies the first part. As described above, an infinite loop space yields a spectrum and therefore a cohomology theory via iterated delooping. If we consider $B\mathrm{Aut}(A\otimes\mathbb{K})$ as the first space of the spectrum, we obtain the 0th one by forming the loop space. But this is

$$\Omega B\mathrm{Aut}(A\otimes\mathbb{K})\simeq\mathrm{Aut}(A\otimes\mathbb{K})\ ,$$

which proves the second statement. The last statement follows from Theorem 2.2.22. □

Corollary 2.3.9. *For any strongly self-absorbing C^*-algebra A the space $B\mathrm{Aut}_0(A\otimes\mathbb{K})$ is an infinite loop space with respect to its natural tensor product operation. The corresponding cohomology theory is denoted by $\bar{E}_A^*(X)$.*

Proof. The proof is entirely similar to the proof of Theorem 2.3.8, except that we replace the category $\mathcal{B}$ by the topological category $\mathcal{B}^0$ which has as its object space just a single point and the group $\mathrm{Aut}_0(A\otimes\mathbb{K})$ as its morphism space. Likewise we replace the category $\mathcal{B}_\otimes$ by the category $\mathcal{B}^0_\otimes$ defined as follows. The object space of $\mathcal{B}^0_\otimes$ is $\mathbb{N}_0$. The morphisms $\hom(n,m)$ are given by those maps α in $\hom((A\otimes\mathbb{K})^{\otimes n},(A\otimes\mathbb{K})^{\otimes m})$ with the property that $[\alpha((1\otimes e)^{\otimes n})]=[(1\otimes e)^{\otimes m}]$ in $K_0((A\otimes\mathbb{K})^{\otimes m})$. The proof of Lemma 2.3.6 still works with the following modifications: There are straightforward replacements $\mathcal{E}^0$, $\left(\mathcal{B}^1_\otimes\right)^0$ and $\mathcal{H}^0$ of the categories $\mathcal{E}$, $\mathcal{B}^1_\otimes$ and $\mathcal{H}$ by taking those endomorphisms that preserve the K-theory class of $1\otimes e$. The isomorphisms ψ_k used in the proof can be chosen such that $\psi_k(1\otimes e)=(1\otimes e)^{\otimes k}$. The restriction functor $\mathcal{H}^0\to\left(\mathcal{B}^1_\otimes\right)^0$ still induces an equivalence by Lemma 2.2.8 and Theorem 2.2.5. □

Remark 2.3.10. We have seen that the classifying space $B\mathrm{Aut}(A \otimes \mathbb{K})$ has the homotopy type of a CW complex. Since its homotopy groups are countable, it follows that this space is homotopy equivalent to a locally finite simplicial complex and hence to an absolute neighborhood extensor, see [93, Thm.6.1, p.137]. It follows that $E^1_A(X)$ is a continuous functor in the sense that if X is the projective limit of projective system $(X_n)_n$ of compact metrizable spaces, then $E^1_A(X) \cong \varinjlim E^1_A(X_n)$, see [60, Thm.11.9, p.287]. Since any compact metrizable space X is the projective limit of a system of finite polyhedra $(X_n)_n$ by [60], one can approach the computation of $E^1_A(X)$ by first computing $E^1_A(X_n)$ using the Atiyah-Hirzebruch spectral sequence.

2.4 A generalized Dixmier-Douady theory

Recall from [54, 10.4] that if $\mathcal{B} = ((B(x))_{x \in X}, \Theta)$ is a continuous field of C^*-algebras over a locally compact space X, the C^*-algebra B associated to $\mathcal{B}$ consists of all elements θ of Θ such that the function $x \mapsto \|\theta(x)\|$ vanishes at infinity. As it has become customary in the literature, the C^*-algebra B will be also referred to as a continuous field of C^*-algebras. Note that $B = \Theta$ if X is compact.

Definition 2.4.1. Let B be a continuous field of C^*-algebras over a locally compact metrizable space X whose fibers are stably isomorphic to strongly self-absorbing C^*-algebras, which are not necessarily mutually isomorphic. We say that B satisfies *the Fell condition* if for each point $x \in X$, there is a closed neighborhood V of x and a projection $p \in B(V)$ such that $[p(v)] \in K_0(B(v))^\times$ for all $v \in V$. If one can choose $V = X$, then we say that B satisfies the global Fell condition.

Theorem 2.4.2. *Let A be a strongly self-absorbing C^*-algebra. Let X be a locally compact space of finite covering dimension and let B be a separable continuous field of C^*-algebras over X with all fibers abstractly isomorphic to $A \otimes \mathbb{K}$. Then B is locally trivial if and only if it satisfies Fell's condition. If X is compact, then B is trivial if and only if B satisfies the global Fell condition.*

Proof. Suppose that there is a projection $p \in B(V)$ such that $[p(v)] \in K_0(B(v))^\times$ for all v in a compact subset V of X. We will show that $B(V) \cong C(V) \otimes A \otimes \mathbb{K}$. First we observe that by Lemma 2.2.14 it follows that $p(v)B(v)p(v) \cong A$, since $B(v) \cong A \otimes \mathbb{K}$. Therefore $pB(V)p$ is a unital continuous field over a finite dimensional space with fibers isomorphic to A and hence $pB(V)p \cong C(V) \otimes A$ by [50]. Second, since p is a full projection, we have that $pB(V)p \otimes \mathbb{K} \cong B(V) \otimes \mathbb{K}$ as $C(V)$-algebras by [24]. Third, $B(V) \otimes \mathbb{K} \cong B(V)$ by [72] since V is finite dimensional and each fiber of B is stable. Putting these facts together we obtain the desired conclusion:

$$B(V) \cong B(V) \otimes \mathbb{K} \cong pB(V)p \otimes \mathbb{K} \cong C(V) \otimes A \otimes \mathbb{K}. \qquad \square$$

Corollary 2.4.3. *Let X be a locally compact space of finite covering dimension. Any separable continuous field of C^*-algebras over X with all fibers abstractly isomorphic to $M_{\mathbb{Q}} \otimes \mathbb{K}$ is locally trivial.*

Proof. Let B be a continuous field as in the statement. In view of Theorem 2.4.2 it suffices to show that B satisfies the Fell condition. Fix $x \in X$ and let $p_0 \in B(x) \cong M_{\mathbb{Q}} \otimes \mathbb{K}$ be a non-zero projection. Since $\mathbb{C}$ is semiprojective, we can lift p_0 to a projection in $A(V)$ for some closed neighborhood V of x. Since A is a continuous field, the map $v \mapsto \|p(v)\|$ is continuous. Thus by shrinking V we can arrange that $p(v) \neq 0$ for all $v \in V$, since $\|p(x)\| = \|p_0\| = 1$. Since $B(v) \cong M_{\mathbb{Q}} \otimes \mathbb{K}$ it follows that $[p(v)] \in K_0(M_{\mathbb{Q}}) \setminus \{0\} \cong \mathbb{Q}^\times \cong K_0(M_{\mathbb{Q}})^\times$. $\square$

Having obtained an efficient criterion for local triviality, we now turn to the question of classifying locally trivial continuous fields of C^*-algebras by cohomological invariants. Let X be a finite connected CW complex. Let $R = K_0(A)$ and let $R_+^\times$ denote the multiplicative abelian group $K_0(A)_+^\times$. If A is purely infinite, then $K_0(A)_+^\times = K_0(A)^\times$ and so $R_+^\times = R^\times$. Suppose that A satisfies the UCT. Then $K_1(A) = 0$ by [139].

The coefficients of the generalized cohomology theory $E_A^*(X)$ were computed in Theorem 2.2.18. Consequently, by [71], the E_2-page of the Atiyah-Hirzebruch spectral sequence for the generalized cohomology $E_A^*(X)$, $A \neq \mathbb{C}$, looks as shown below.

	0	1	2	3
0	$H^0(X, R_+^\times)$	$H^1(X, R_+^\times)$	$H^2(X, R_+^\times)$	$H^3(X, R_+^\times)$
-1	0	0	0	0
-2	$H^0(X, R)$	$H^1(X, R)$	$H^2(X, R)$	$H^3(X, R)$
-3	0	0	0	0
-4	$H^0(X, R)$	$H^1(X, R)$	$H^2(X, R)$	$H^3(X, R)$

If $A = \mathbb{C}$, all the rows of the E_2-page of $E_{\mathbb{C}}^1(X)$ are null with the exception of the (-2)-row whose entries are $H^p(X, \mathbb{Z})$, $p \geq 0$. Since the differentials in the Atiyah-Hirzebruch spectral sequence are torsion operators, [7, Thm.2.7], we obtain the following.

Corollary 2.4.4. *Let X be a finite connected CW complex such that $H^*(X,R)$ is torsion free. If $A \neq \mathbb{C}$ satisfies the UCT, then*

$$\mathcal{B}un_X(A \otimes \mathbb{K}) \cong E^1_A(X) \cong H^1(X, R^\times_+) \times \prod_{k \geq 1} H^{2k+1}(X, R) \ .$$

Corollary 2.4.5. *Let X be a compact metrizable space and let $M_\mathbb{Q}$ denote the universal UHF-algebra with $K_0(M_\mathbb{Q}) \cong \mathbb{Q}$. Then there are natural isomorphism of groups*

$$\mathcal{B}un_X(M_\mathbb{Q} \otimes \mathbb{K}) \cong E^1_{M_\mathbb{Q}}(X) \cong H^1(X, \mathbb{Q}^\times_+) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \ .$$

$$\mathcal{B}un_X(M_\mathbb{Q} \otimes \mathcal{O}_\infty \otimes \mathbb{K}) \cong E^1_{M_\mathbb{Q} \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \ .$$

Proof. Set $h^*(X) = E^*_A(X)$, $\bar{h}^*(X) = \bar{E}^*_A(X)$ (see Cor. 2.3.9) and $R = K_0(A)$ where A is either $M_\mathbb{Q}$ or $M_\mathbb{Q} \otimes \mathcal{O}_\infty$. We will show that there are natural isomorphisms (i) $h^1(X) \cong H^1(X, R^\times_+) \oplus \bar{h}^1(X)$ and (ii) $\bar{h}^1(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$. Note that $\bar{h}^*(pt) = t\,\mathbb{Q}[t]$ with $deg(t) = -2$ and $h^*(pt) = R^\times_+ \oplus t\,\mathbb{Q}[t]$. Suppose first that X is a finite connected CW-complex. Then (ii) follows by applying the isomorphism established in equation (3.20) of [71, p.48] since $\bar{h}^*(pt)$ is a vector spaces over $\mathbb{Q}$. If G is a topological group and H a normal subgroup of G such that $H \to G \to G/H$ is a principal H-bundle, then there is a homotopy fibre sequence $G \to G/H \to BH \to BG \to B(G/H)$ and hence an exact sequence of pointed sets $[X, G] \to [X, G/H] \to [X, BH] \to [X, BG] \to [X, B(G/H)]$. Using this for the principal bundle from Corollary 2.2.19 we obtain an exact sequence of groups $0 \to \bar{h}^1(X) \to h^1(X) \xrightarrow{\delta_0} H^1(X, R^\times_+)$. We want to compare this sequence with the exact sequence $0 \to F^2h^1(X) \to h^1(X) \to H^1(X, R^\times_+) \to 0$ given by the Atiyah-Hirzebruch spectral sequence. Recall that $F^2h^1(X) = \ker(h^1(X) \to h^1(X_1))$, where X_1 is the 1-skeleton of X. Since both maps with target $H^1(X_1, R^\times_+)$ are injective in the following commutative diagram induced by $X_1 \hookrightarrow X$

$$\begin{tikzcd}
h^1(X) \arrow[r, "\delta_0"] \arrow[d] & H^1(X, R^\times_+) \arrow[d] \\
h^1(X_1) \arrow[r] & H^1(X_1, R^\times_+)
\end{tikzcd}$$

we deduce that $F^2h^1(X) \cong \ker(\delta_0) \cong \bar{h}^1(X)$ and hence obtain an exact sequence $0 \to \bar{h}^1(X) \to h^1(X) \to H^1(X, R^\times_+) \to 0$. Since $\bar{h}^1(X)$ is a divisible group it follows that $h^1(X)$ splits as $H^1(X, R^\times_+) \oplus \bar{h}^1(X)$. To verify that there is a natural splitting one employs the natural transformation $h^*(X) \to \bar{h}^*(X)$ induced by the coefficient map $h^*(pt) \to \bar{h}^*(pt)$,

$(r, f(t)) \mapsto f(t)$, see [71, Thm.3.22].

For the general case we write X as a projective limit of a system of polyhedra $(X_n)_n$ and then we apply the continuity property of $E^1_A(X)$ as discussed in Remark 2.3.10. $\square$

Let $A \not\cong \mathcal{O}_2$ be a strongly self-absorbing C^*-algebra that satisfies the UCT. Then $A \otimes M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty$ by [139]. The canonical unital embedding $A \to A \otimes M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty$ induces a morphism of groups $\mathrm{Aut}(A \otimes \mathbb{K}) \to \mathrm{Aut}(M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \otimes \mathbb{K})$ and hence a morphism of groups

$$\delta : E^1_A(X) \to E^1_{M_{\mathbb{Q}} \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}).$$

Definition 2.4.6. We define rational characteristic classes $\delta_0 : \mathcal{B}un_X(A \otimes \mathbb{K}) \to H^1(X, \mathbb{Q}^\times)$ and $\delta_k : \mathcal{B}un_X(A \otimes \mathbb{K}) \to H^{2k+1}(X, \mathbb{Q})$, $k \geq 1$, to be the components of the map δ from above. It is clear that δ_0 is lifts to a map $\delta_0 : \mathcal{B}un_X(A \otimes \mathbb{K}) \to H^1(X, K_0(A)^\times_+)$ induced by the morphism of groups $\mathrm{Aut}(A \otimes \mathbb{K}) \to \pi_0(\mathrm{Aut}(A \otimes \mathbb{K})) \cong K_0(A)^\times_+$ and which gives the obstruction to reducing the structure group to $\mathrm{Aut}_0(A \otimes \mathbb{K})$. We will see in Corollary 2.4.8 that δ_1 also lifts to an integral class with values in $H^3(X, \mathbb{Z})$ for $A = \mathcal{Z}$. One has $\delta_k(B_1 \otimes B_2) = \delta_k(B_1) + \delta_k(B_2)$, $k \geq 0$.

Since the differentials in the Atiyah-Hirzebruch spectral sequence are torsion operators we deduce:

Corollary 2.4.7. *Let A be a strongly self-absorbing C^*-algebra that satisfies the UCT. Let X be a finite connected CW complex such that $H^*(X, \mathbb{Z})$ is torsion free. Then:*

(i) $B_1, B_2 \in \mathcal{B}un_X(A \otimes \mathbb{K})$ are isomorphic if and only $\delta_k(B_1) = \delta_k(B_2)$ for all $k \geq 0$.

(ii) $\mathcal{B}un_X(\mathcal{Z} \otimes \mathbb{K}) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z})$ and

$$\mathcal{B}un_X(\mathcal{O}_\infty \otimes \mathbb{K}) \cong H^1(X, \mathbb{Z}/2) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z}) \ .$$

Corollary 2.4.8. *Let X be a compact connected metrizable space. Let A be a strongly self-absorbing C^*-algebra, which satisfies the UCT. Then $H^{i+2}(X, K_0(A))$ is a natural direct summand of $\bar{E}^i_A(X)$ for all $i \geq 0$. It follows that there is a natural homomorphism*

$$\bar{\delta}_1 \colon \bar{E}^1_A(X) \to H^3(X, K_0(A))$$

giving back the usual Dixmier-Douady class for $A = \mathbb{C}$.

Proof. Recall that $E^i_{\mathbb{C}}(X) \cong \bar{E}^i_{\mathbb{C}}(X) \cong H^{i+2}(X, \mathbb{Z})$. Using the continuity properties discussed in Remark 2.3.10, we may assume that X is a finite connected CW complex. Since A satisfies the UCT, $K_1(A) = 0$ and $K_0(A) \subset \mathbb{Q}$ is flat and satisfies $K_0(A) \otimes K_0(A) \cong K_0(A)$. The natural transformation of cohomology theories $\bar{E}^*_A(X) \xrightarrow{\cong} \bar{E}^*_A(X) \otimes K_0(A)$ is an isomorphism since it is so on coefficients. The unital map $\mathbb{C} \to A$ induces a natural transformation of cohomology theories $T\colon \bar{E}^*_{\mathbb{C}}(X) \otimes K_0(A) \to \bar{E}^*_A(X) \otimes K_0(A) \cong \bar{E}^*_A(X)$. The desired conclusion follows now from the naturality of the Atiyah-Hirzebruch spectral sequence since all the rows of the E_2-page of $\bar{E}^*_{\mathbb{C}}(X) \otimes K_0(A)$ are null with the exception of the (-2)-row and T induces the identity map on this row due to the isomorphism $\pi_2(\mathrm{Aut}(\mathbb{K})) \otimes K_0(A) \to \pi_2(\mathrm{Aut}(A \otimes \mathbb{K})) \otimes K_0(A) \cong \pi_2(\mathrm{Aut}(A \otimes \mathbb{K}))$, see Theorem 2.2.18. The edge homomorphism $\bar{E}^i_A(X) \to H^{i+2}(X, K_0(A))$ and T give the splitting. $\square$

Corollary 2.4.9. *Let X be a compact metrizable space and let A be a strongly self-absorbing C^*-algebra. Two bundles $B_1, B_2 \in \mathcal{B}un_X(A \otimes \mathbb{K})$ are isomorphic if and only if $B_1 \otimes \mathcal{O}_\infty \cong B_2 \otimes \mathcal{O}_\infty$.*

Proof. Without any loss of generality we may assume that X is a finite CW-complex. By Corollary 2.2.19 there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}_0(A \otimes \mathbb{K}) \arrow[r] \arrow[d] & \mathrm{Aut}(A \otimes \mathbb{K}) \arrow[r] \arrow[d] & K_0(A)^\times_+ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Aut}_0(\mathcal{O}_\infty \otimes A \otimes \mathbb{K}) \arrow[r] & \mathrm{Aut}(\mathcal{O}_\infty \otimes A \otimes \mathbb{K}) \arrow[r] & K_0(A)^\times \arrow[r] & 0 \end{tikzcd}$$

Passing to classifying spaces we obtain a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & {[X, B\mathrm{Aut}_0(A \otimes \mathbb{K})]} \arrow[r] \arrow[d, "i"'] & {[X, B\mathrm{Aut}(A \otimes \mathbb{K})]} \arrow[r] \arrow[d, "T"'] & H^1(X, K_0(A)^\times_+) \arrow[d, "j"'] \\ 0 \arrow[r] & {[X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes A \otimes \mathbb{K})]} \arrow[r] & {[X, B\mathrm{Aut}(\mathcal{O}_\infty \otimes A \otimes \mathbb{K})]} \arrow[r] & H^1(X, K_0(A)^\times) \end{tikzcd}$$

This is a diagram of abelian groups by Theorem 2.3.8 and Corollary 2.3.9. The map j is injective. This follows from the exactness of the sequence $H^0(X, R_2) \to H^0(X, R_2/R_1) \to H^1(X, R_1) \to H^1(X, R_2)$ induced by an inclusion of discrete abelian groups $R_1 \hookrightarrow R_2$. Let us argue that the map i is also injective. If $A \neq \mathbb{C}$, this follows from Corollary 2.2.21, whereas for $A = \mathbb{C}$ this was proved in Corollary 2.4.8. The five lemma implies now that the map $T : E^1_A(X) \to E^1_{A \otimes \mathcal{O}_\infty}(X)$ is injective. $\square$

Finally we address the question to what extent the results (i) and (ii) of Dixmier and Douady mentioned in the introduction admit generalizations to our context. The following statement corresponds to (i) and the first part of (ii).

Corollary 2.4.10. *Let B be a separable continuous field of C^*-algebras over a compact metrizable space whose fibers are Morita equivalent to the same strongly self-absorbing C^*-algebra A. Suppose that B satisfies Fell's condition. Then for each $x \in X$, there is a closed neighborhood V of x with the following property. There exists a unital separable continuous field D over V with fibers isomorphic to A such that $B(V) \otimes \mathbb{K} \cong D \otimes \mathbb{K}$. If A is finite dimensional or if X is finite dimensional, then $B \otimes \mathbb{K}$ is locally trivial and therefore we can associate to it an invariant $\delta(B) \in E^1_A(X)$ which classifies $B \otimes \mathbb{K}$ up to isomorphism of continuous fields, and B up to Morita equivalence over X.*

Proof. Let $x \in X$. Let p and V be as in Definition 2.4.1. Letting $D := pB(V)p$ we have already seen in the proof of Theorem 2.4.2 that $B(V) \otimes \mathbb{K} \cong D \otimes \mathbb{K}$ and that all the fibers of D are isomorphic to A. If A is finite dimensional, then $A = \mathbb{C}$, and so obviously $D = C(V)$. This corresponds to the result (i) of Dixmier and Douady. If X is finite dimensional, then $B(V) \otimes \mathbb{K} \cong C(V) \otimes A \otimes \mathbb{K}$ by Theorem 2.4.2. We conclude the proof by applying Theorem 2.3.8. $\square$

As we have just seen, the class $\delta(B) \in E^1_A(X)$ is defined for continuous fields with fibers Morita equivalent to A which satisfy Fell's condition. Furthermore, one can associate rational characteristic classes to certain continuous fields which are typically very far from being locally trivial and whose fibers are not necessarily Morita equivalent to each other.

Corollary 2.4.11. *Let B be a separable continuous field of C^*-algebras over a finite dimensional compact metrizable space whose fibers are Morita equivalent to (possibly different) strongly self-absorbing C^*-algebras satisfying the UCT. Suppose that for each $x \in X$ there is a closed neighborhood V of x and a projection $p \in B(V)$ such that $[p(v)] \neq 0$ in $K_0(B(v))$ for all $v \in V$. Then $B_\sharp := B \otimes \mathcal{O}_\infty \otimes M_\mathbb{Q} \otimes \mathbb{K}$ is locally trivial and so we can associate "stable" rational characteristic classes to B, by defining $\delta_k^{stable}(B) = \delta_k(B_\sharp) \in H^{2k+1}(X, \mathbb{Q})$. These cohomology classes determine $B_\sharp$ up to an isomorphism.*

Proof. The fibers $B_\sharp(x) = B(x) \otimes \mathcal{O}_\infty \otimes M_\mathbb{Q} \otimes \mathbb{K}$ satisfy the UCT and they are stabilized strongly self-absorbing Kirchberg algebras not isomorphic to $\mathcal{O}_2 \otimes \mathbb{K}$ since $K_0(B(x)) \neq 0$. It follows by [139] that they are all isomorphic to $\mathcal{O}_\infty \otimes M_\mathbb{Q} \otimes \mathbb{K}$ and moreover that the induced map $K_0(B(x)) \to K_0(B_\sharp(x))$ is injective for all $x \in X$. It follows that $B_\sharp$ satisfies Fell's condition and hence it is locally trivial by Theorem 2.4.2. We conclude the proof by applying Corollary 2.4.5. $\square$

3 Unit spectra of K-theory from strongly self-absorbing C^*-algebras

We give an operator algebraic model for the first group of the unit spectrum $gl_1(KU)$ of complex topological K-theory, i.e. $[X, BGL_1(KU)]$, by bundles of stabilized infinite Cuntz C^*-algebras $\mathcal{O}_\infty \otimes \mathbb{K}$. We develop similar models for the localizations of KU at a prime p and away from p. Our work is based on the $\mathcal{I}$-monoid model for the units of K-theory by Sagave and Schlichtkrull and it was motivated by the goal of finding connections between the infinite loop space structure of the classifying space of the automorphism group of stabilized strongly self-absorbing C^*-algebras that arose in our generalization of the Dixmier-Douady theory and classical spectra from algebraic topology.

3.1 Introduction

Suppose $E^\bullet$ is a multiplicative generalized cohomology theory represented by a commutative ring spectrum R. The units $GL_1(E^0(X))$ of $E^0(X)$ provide an abelian group functorially associated to the space X. From the point of view of algebraic topology it is therefore a natural question, whether we can lift GL_1 to spectra, i.e. whether there is a spectrum of units $gl_1(R)$ such that $gl_1(R)^0(X) = GL_1(E^0(X))$.

It was realized by Sullivan in [133] that $gl_1(R)$ is closely connected to questions of orientability in algebraic topology. In particular, the units of K-theory act on the K-orientations of PL-bundles. Segal [129] proved that the classifying space $\{1\} \times BU \subset \mathbb{Z} \times BU$ for virtual vector bundles of virtual dimension 1 equipped with the H-space structure from the tensor product is in fact a Γ-space, which in turn yields a spectrum of a connective generalized cohomology theory $bu^*_\otimes(X)$. His method is easily extended to include the virtual vector bundles of virtual dimension -1 to obtain a generalized cohomology theory $gl_1(KU)^*(X) \supset bu^*_\otimes(X)$ answering the above question affirmatively: $GL_1(K^0(X)) \cong gl_1(KU)^0(X)$. Later May, Quinn, Ray and Tornehave [104] came up with the notion of E_∞-ring spectra, which always have associated unit spectra.

Since $gl_1(R)$ is defined via stable homotopy theory, there is in general no nice geometric interpretation of the higher groups even though R may have one. In particular, no geometric interpretation was known for $gl_1(KU)^k(X)$. In this article we give an operator algebra interpretation of $gl_1(KU)^1(X)$ as the group of isomorphism classes of locally trivial bundles of C*-algebras with fiber isomorphic to the stable Cuntz algebra $\mathcal{O}_\infty \otimes \mathbb{K}$ with the group operation induced by the tensor product. In fact one can also recover Segal's original infinite loop space $BBU_\otimes$ as $B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})$, where $\mathcal{Z}$ is the ubiquitous Jiang-Su algebra [139]. For localizations of KU we obtain that $gl_1(KU_{(p)})^1(X)$ is the group of isomorphism classes of locally trivial bundles with fiber isomorphic to the C*-algebra $M_{(p)} \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ with the group operation induced by the tensor product. Here $M_{(p)}$ is a C^*-algebra with $K_0(M_{(p)}) \cong \mathbb{Z}_{(p)}$, $K_1(M_{(p)}) = 0$ that can be obtained as an infinite tensor product of matrix algebras.

Our approach is based on the work of Sagave and Schlichtkrull [123, 124], who developed a representation of $gl_1(R)$ for a commutative symmetric ring spectrum R as a commutative $\mathcal{I}$-monoid. Motivated by the definition of twisted cohomology theories, we study the following situation, which appears to be a natural setup beyond the case where R is K-theory: Suppose G is an $\mathcal{I}$-space, such that each $G(\mathbf{n})$ is a topological group acting on R_n. To formulate a sensible compatibility condition between the group action κ and the multiplication μ^R on R, we need to demand that G itself carries an additional $\mathcal{I}$-monoid structure μ^G and the following diagram commutes:

$$\begin{tikzcd}
G(\mathbf{m}) \times R_m \times G(\mathbf{n}) \times R_n \arrow[r, "\kappa_m \times \kappa_n"] \arrow[d, "(\mu^G_{m,n} \times \mu^R_{m,n}) \circ \tau"'] & R_m \times R_n \arrow[d, "\mu^R_{m,n}"] \\
G(\mathbf{m} \sqcup \mathbf{n}) \times R_{m+n} \arrow[r, "\kappa_{m+n}"'] & R_{m+n}
\end{tikzcd}$$

where τ switches the two middle factors. Associativity of the group action suggests that the analogous diagram, which has $G(\mathbf{n})$ in place of R_n and μ^G instead of μ^R, should also commute. This condition can be seen as a homotopy theoretic version of the property needed for the Eckmann-Hilton trick, which is why we will call such a G an Eckmann-Hilton $\mathcal{I}$-group (EH-$\mathcal{I}$-group for short). Commutativity of the above diagram has the following important implications:

- the $\mathcal{I}$-monoid structure of G is commutative (Lemma 3.3.2),
- the classifying spaces $B_\nu G(\mathbf{n})$ with respect to the group multiplication ν of G form a commutative $\mathcal{I}$-monoid $\mathbf{n} \mapsto B_\nu G(\mathbf{n})$,
- if G is convergent and $G(\mathbf{n})$ has the homotopy type of a CW-complex, then the Γ-spaces associated to G and $B_\nu G$ satisfy $B_\mu \Gamma(G) \simeq \Gamma(B_\nu G)$, where $B_\mu \Gamma(X)$ for a commutative $\mathcal{I}$-monoid X denotes the Γ-space delooping of $\Gamma(X)$ (Theorem 3.3.6).

Let $\Omega^\infty(R)^*(\mathbf{n})$ be the commutative $\mathcal{I}$-monoid with associated spectrum $gl_1(R)$. If G acts on R and the inverses of G with respect to both multiplicative structures μ^G and ν are compatible in the sense of Definition 3.3.1, then the action induces a map of Γ-spaces $\Gamma(G) \to \Gamma(\Omega^\infty(R)^*)$. This deloops to a map $B_\mu\Gamma(G) \to B_\mu\Gamma(\Omega^\infty(R)^*)$ and we give sufficient conditions for this to be a strict equivalence of (very special) Γ-spaces.

In the second part of the paper, we consider the EH-$\mathcal{I}$-group $G_A(\mathbf{n}) = \mathrm{Aut}((A \otimes \mathbb{K})^{\otimes n})$ associated to the automorphisms of a (stabilized) strongly self-absorbing C^*-algebra A. This class of C^*-algebras was introduced by Toms and Winter in [139]. It contains the algebras $\mathcal{O}_\infty$ and $M_{(p)}$ alluded to above as well as the Jiang-Su algebra $\mathcal{Z}$ and the Cuntz algebra $\mathcal{O}_2$. It is closed with respect to tensor product and plays a fundamental role in the classification theory of nuclear C^*-algebras.

For a strongly self-absorbing C^*-algebra A, $X \mapsto K_0(C(X) \otimes A)$ turns out to be a multiplicative cohomology theory. In fact, this structure can be lifted to a commutative symmetric ring spectrum KU^A along the lines of [69, 79, 53]. The authors showed in [47] that $B\mathrm{Aut}(A \otimes \mathbb{K})$ is an infinite loop space and the first space in the spectrum of a generalized cohomology theory $E^*_A(X)$ such that $E^0_A(X) \cong K_0(C(X) \otimes A)^\times_+$, in particular $E^0_{\mathcal{O}_\infty}(X) \cong GL_1(K^0(X))$, which suggests that $E^*_{\mathcal{O}_\infty}(X) \cong gl_1(KU)^*(X)$. In fact, we can prove:

Theorem 3.1.1. *Let $A \neq \mathbb{C}$ be a separable strongly self-absorbing C^*-algebra.*

(a) The EH-$\mathcal{I}$-group G_A associated to A acts on the commutative symmetric ring spectrum $KU^A_\bullet$ inducing a map $\Gamma(G_A) \to \Gamma(\Omega^\infty(KU^A)^*)$.

(b) The induced map on spectra is an isomorphism on all homotopy groups π_n with $n > 0$ and the inclusion $K_0(A)^\times_+ \to K_0(A)^\times$ on π_0.

(c) In particular $B\mathrm{Aut}(A \otimes \mathcal{O}_\infty \otimes \mathbb{K}) \simeq BGL_1(KU^A)$ and $gl_1(KU)^1(X) \cong \mathcal{B}un(A \otimes \mathcal{O}_\infty \otimes \mathbb{K})$, where the right hand side denotes the group of isomorphism classes of C^*-algebra bundles with fiber $A \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ with respect to the tensor product.

We also compare the spectrum defined by the Γ-space $\Gamma(B_\nu G_A)$ with the one obtained from the infinite loop space construction used in [47] and show that they are equivalent. The group $gl_1(KU^A)^1(X)$ alias $E^1_A(X)$ is a natural receptacle for invariants of not necessarily locally trivial continuous fields of C^*-algebras with stable strongly self-absorbing fibers that satisfy a Fell condition. This provides a substantial extension of results by Dixmier and Douady with $gl_1(KU^A)$ replacing ordinary cohomology, [47]. The above theorem lays the ground for an operator algebraic interpretation of the "higher" twists of K-theory. Twisted K-theory as defined first by Donovan and Karoubi [56] and later in increased generality by Rosenberg [122] and Atiyah and Segal [8] has a nice interpretation in terms of bundles of compact operators [122]. From the point of view of homotopy theory, it is possible to define twisted K-theory with more than just the $K(\mathbb{Z}, 3)$-twists [5, 102] and the present paper suggests an interpretation of these more general invariants in terms of bundles with fiber

$\mathcal{O}_\infty \otimes \mathbb{K}$. We will pursue this idea in upcoming work.

3.2 Preliminaries

3.2.1 Symmetric ring spectra, units and $\mathcal{I}$-spaces

Since our exposition below is based on symmetric ring spectra and their units, we will recall their definition in this section. The standard references for this material are [74] and [96]. Let Σ_n be the symmetric group on n letters and let $S^n = S^1 \wedge \cdots \wedge S^1$ be the smash product of n circles. This space carries a canonical Σ_n-action. Define S^0 to be the two-point space. Let $\mathcal{T}op$ be the category of compactly generated Hausdorff spaces and denote by $\mathcal{T}op_*$ its pointed counterpart.

Definition 3.2.1. A *commutative symmetric ring spectrum* $R_\bullet$ consists of a sequence of pointed topological spaces R_n for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ with a basepoint preserving action by Σ_n together with a sequence of pointed equivariant maps $\eta_n \colon S^n \to R_n$ and a collection $\mu_{m,n}$ of pointed $\Sigma_m \times \Sigma_n$-equivariant maps

$$\mu_{m,n} \colon R_m \wedge R_n \to R_{m+n}$$

such that the following conditions hold:

(a) associativity: $\mu_{p+q,r} \circ (\mu_{p,q} \wedge \mathrm{id}_{R_r}) = \mu_{p,q+r} \circ (\mathrm{id}_{R_p} \wedge \mu_{q,r})$,

(b) compatibility: $\mu_{p,q} \circ (\eta_p \wedge \eta_q) = \eta_{p+q}$ and $\mu_{p,0} \circ (\mathrm{id}_{R_p} \wedge \eta_0) = \mathrm{id}_{R_p}$, $\mu_{0,q} \circ (\eta_0 \wedge \mathrm{id}_{R_q}) = \mathrm{id}_{R_q}$ where we identify $S^0 \wedge R_p$ and $R_p \wedge S^0$ with R_p via the canonical maps,

(c) commutativity: The following diagram commutes

$$\begin{tikzcd} R_m \wedge R_n \arrow[r, "\mu_{m,n}"] \arrow[d, "\mathrm{tw}"'] & R_{m+n} \arrow[d, "\tau_{m,n}"] \\ R_n \wedge R_m \arrow[r, "\mu_{n,m}"'] & R_{n+m} \end{tikzcd}$$

where tw is the flip map and $\tau_{m,n}$ is the block permutation exchanging the first m letters with the last n letters preserving their order.

In order to talk about units in a symmetric ring spectrum with respect to its graded multiplication $\mu_{m,n} \colon R_m \wedge R_n \to R_{m+n}$ we need to deal with homotopy colimits. Given a small (discrete) indexing category $\mathcal{J}$ and a functor $F \colon \mathcal{J} \to \mathcal{T}op$, i.e. a diagram in spaces, we define its *homotopy colimit* $\mathrm{hocolim}_{\mathcal{J}} F$ to be the geometric realization of its (topological) *transport category* $\mathcal{T}_F$ with object space $\coprod_{j\in\mathrm{obj}(\mathcal{J})} F(j)$ and morphism space $\coprod_{j,j'\in\mathrm{obj}(\mathcal{J})} F(j)\times\hom_{\mathcal{J}}(j,j')$, where the source map is given by the projection to the first factor and the value of the target map on a morphism $(x,f)\in F(j)\times\hom_{\mathcal{J}}(j,j')$ is $F(f)(x)$ [147, Proposition 5.7].

To define inverses for the graded multiplication μ of R, we need a bookkeeping device that keeps track of the degree, i.e. the suspension coordinate. We follow the work of Sagave and Schlichtkrull [123, 124], in particular [124, section 2.2], which tackles this issue using $\mathcal{I}$-spaces and $\mathcal{I}$-monoids.

Definition 3.2.2. Let $\mathcal{I}$ be the category, whose objects are the finite sets $\mathbf{n}=\{1,\dots,n\}$ and whose morphisms are injective maps. The empty set $\mathbf{0}$ is an initial object in this category. Concatenation $\mathbf{m}\sqcup\mathbf{n}$ and the block permutations $\tau_{m,n}\colon \mathbf{m}\sqcup\mathbf{n}\to\mathbf{n}\sqcup\mathbf{m}$ turn $\mathcal{I}$ into a symmetric monoidal category. An *$\mathcal{I}$-space* is a functor $X\colon\mathcal{I}\to\mathcal{T}op_*$.

Moreover, an $\mathcal{I}$-space X is called an *$\mathcal{I}$-monoid* if it comes equipped with a natural transformation $\mu\colon X\times X\to X\circ\sqcup$ of functors $\mathcal{I}^2\to\mathcal{T}op_*$, i.e. a family of continuous maps

$$\mu_{m,n}\colon X(\mathbf{m})\times X(\mathbf{n})\to X(\mathbf{m}\sqcup\mathbf{n})\ ,$$

which is associative in the sense that $\mu_{l,m+n}\circ(\mathrm{id}_{X(\mathbf{l})}\times\mu_{m,n})=\mu_{l+m,n}\circ(\mu_{l,m}\times\mathrm{id}_{X(\mathbf{n})})$ for all $\mathbf{l},\mathbf{m},\mathbf{n}\in\mathrm{obj}(\mathcal{I})$ and unital in the sense that the diagrams

$$\begin{tikzcd} X(\mathbf{0})\times X(\mathbf{n}) \arrow[r, "\mu_{0,n}"] & X(\mathbf{n}) \\ X(\mathbf{n}) \arrow[u] \arrow[ur, "\mathrm{id}_{X(\mathbf{n})}"'] & \end{tikzcd} \qquad \begin{tikzcd} X(\mathbf{n})\times X(\mathbf{0}) \arrow[r, "\mu_{n,0}"] & X(\mathbf{n}) \\ X(\mathbf{n}) \arrow[u] \arrow[ur, "\mathrm{id}_{X(\mathbf{n})}"'] & \end{tikzcd}$$

commute for every $\mathbf{n}\in\mathrm{obj}(\mathcal{I})$, where the two upwards arrows are the inclusion with respect to the basepoint in $X(\mathbf{0})$. Likewise we call an $\mathcal{I}$-monoid X *commutative*, if

$$\begin{tikzcd} X(\mathbf{m})\times X(\mathbf{n}) \arrow[r, "\mu_{m,n}"] \arrow[d, "\mathrm{sw}"'] & X(\mathbf{m}\sqcup\mathbf{n}) \arrow[d, "\tau_{m,n*}"] \\ X(\mathbf{n})\times X(\mathbf{m}) \arrow[r, "\mu_{n,m}"] & X(\mathbf{n}\sqcup\mathbf{m}) \end{tikzcd}$$

commutes. We denote the homotopy colimit of X over $\mathcal{I}$ by $X_{h\mathcal{I}}=\mathrm{hocolim}_{\mathcal{I}}(X)$. If X is an $\mathcal{I}$-monoid, then $(X_{h\mathcal{I}},\mu)$ is a topological monoid as explained in [124, p. 652]. We call X *grouplike*, if $\pi_0(X_{h\mathcal{I}})$ is a group with respect to the multiplication induced by the monoid structure.

Let $\mathcal{N}$ be the category associated to the directed poset $\mathbb{N}_0=\{0,1,2,...\}$. Note that there is an inclusion functor $\mathcal{N} \to \mathcal{I}$, which sends a map $n \to m$ to the standard inclusion $\mathbf{n} \to \mathbf{m}$. This way, we can associate a space called the *telescope* to an $\mathcal{I}$-space X: $\mathrm{Tel}(X) := \mathrm{hocolim}_{\mathcal{N}} X$. If X is convergent, then $\mathrm{Tel}(X) \to X_{h\mathcal{I}}$ is a weak homotopy equivalence. Any space Y together with a continuous self-map $f \colon Y \to Y$ yields a functor $F \colon \mathcal{N} \to \mathcal{T}op$ with $F(n) = Y$, $F(n \to m) = f^{(m-n)}$, where $f^{(m-n)}$ denotes the composition of $m-n$ copies of f. We will denote the associated telescope by $\mathrm{Tel}(Y; f)$ or $\mathrm{Tel}(Y)$ if the map is clear.

As described in [124, section 2.3], a symmetric ring spectrum $R_\bullet$ yields an $\mathcal{I}$-monoid $\Omega^\infty(R)$ as follows: Let $\Omega^\infty(R)(\mathbf{n}) = \Omega^n(R_n)$ with basepoint η_n. A morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ uniquely defines a permutation $\bar{\alpha} \colon \mathbf{n} = \mathbf{l} \sqcup \mathbf{m} \to \mathbf{n}$, which is order preserving on the first l elements and given by α on the last m entries. Mapping $f \in \Omega^m(R_m)$ to

$$S^n \xrightarrow{\bar{\alpha}^{-1}} S^n \xrightarrow{\eta_l \wedge f} R_l \wedge R_m \xrightarrow{\mu_{l,m}} R_n \xrightarrow{\bar{\alpha}} R_n \ .$$

yields the functoriality with respect to injective maps. The monoid structure is induced by the multiplication of $R_\bullet$ as follows

$$\mu_{m,n}(f,g) \colon \ S^m \wedge S^n \xrightarrow{f \wedge g} R_m \wedge R_n \xrightarrow{\mu_{m,n}} R_{m+n}$$

for $f \in \Omega^m(R_m)$ and $g \in \Omega^n(R_n)$. If R is commutative, then $\Omega^\infty(R)$ is a commutative $\mathcal{I}$-monoid.

Let $\Omega^\infty(R)^*$ be the $\mathcal{I}$-monoid of units of R given as follows $\Omega^\infty(R)^*(\mathbf{n})$ is the union of those components of $\Omega^\infty(R)(\mathbf{n})$ that have stable inverses in the sense that for each $f \in \Omega^\infty(R)^*(\mathbf{n})$ there exists $g \in \Omega^\infty(R)(\mathbf{m})$, such that $\mu_{n,m}(f,g)$ and $\mu_{m,n}(g,f)$ are homotopic to the basepoint of $\Omega^\infty(R)(\mathbf{n} \sqcup \mathbf{m})$ and $\Omega^\infty(R)(\mathbf{m} \sqcup \mathbf{n})$ respectively. Define the space of units by $GL_1(R) = \mathrm{hocolim}_{\mathcal{I}}(\Omega^\infty(R)^*)$.

If R is commutative, the spectrum of units associated to the Γ-space $\Gamma(\Omega^\infty(R)^*)$ will be denoted by $gl_1(R)$. If R is convergent, then $\pi_0((\Omega^\infty(R)^*)_{h\mathcal{I}}) = GL_1(\pi_0(R))$, $\Gamma(\Omega^\infty(R)^*)$ is very special and $gl_1(R)$ is an Ω-spectrum.

3.3 Eckmann-Hilton $\mathcal{I}$-groups

As motivated in the introduction, we study the following particularly nice class of $\mathcal{I}$-monoids.

Definition 3.3.1. Let $\mathcal{G}rp_*$ be the category of topological groups (which we assume to be *well-pointed* by the identity element) and continuous homomorphisms. A functor $G \colon \mathcal{I} \to \mathcal{G}rp_*$ is called an *$\mathcal{I}$-group*. An $\mathcal{I}$-group G is called an *Eckmann-Hilton $\mathcal{I}$-group* (or EH-$\mathcal{I}$-group for short) if it is an $\mathcal{I}$-monoid in $\mathcal{G}rp_*$ with multiplication $\mu_{m,n}$, such that the following diagram of natural transformations between functors $\mathcal{I}^2 \to \mathcal{G}rp_*$ commutes,

$$\begin{tikzcd}
G(\mathbf{m}) \times G(\mathbf{m}) \times G(\mathbf{n}) \times G(\mathbf{n}) \arrow[r, "(\mu_{m,n}\times\mu_{m,n})\circ\tau"] \arrow[d, "\nu_m\times\nu_n"'] & G(\mathbf{m}\sqcup\mathbf{n}) \times G(\mathbf{m}\sqcup\mathbf{n}) \arrow[d, "\nu_{m+n}"] \\
G(\mathbf{m}) \times G(\mathbf{n}) \arrow[r, "\mu_{m,n}"'] & G(\mathbf{m}\sqcup\mathbf{n})
\end{tikzcd}$$

(where $\nu_n\colon G(\mathbf{n}) \times G(\mathbf{n}) \to G(\mathbf{n})$ denotes the group multiplication and τ is the map that switches the two innermost factors). We call G *convergent*, if it is convergent as an $\mathcal{I}$-space in the sense of [124, section 2.2]. If all morphisms in $\mathcal{I}$ except for the maps $\mathbf{0} \to \mathbf{n}$ are mapped to homotopy equivalences, the EH-$\mathcal{I}$-group G is called *stable* (this implies convergence).

Let $\iota_m\colon \mathbf{0} \to \mathbf{m}$ be the unique morphism. We say that an EH-$\mathcal{I}$-group has *compatible inverses*, if there is a path from $(\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g) \in G(\mathbf{m}\sqcup\mathbf{m})$ to $(\mathrm{id}_{\mathbf{m}} \sqcup \iota_m)_*(g)$ for all $\mathbf{m} \in \mathrm{obj}(\mathcal{I})$ and $g \in G(\mathbf{m})$.

The above diagram is easily recognized as a graded version of the Eckmann-Hilton compatibility condition, where the group multiplication and the monoid structure provide the two operations. Thus, the following comes as no surprise.

Lemma 3.3.2. *Let G be an EH-$\mathcal{I}$-group. Then the $\mathcal{I}$-monoid structure of G is commutative.*

Proof. Let $1_m \in G(\mathbf{m})$ be the identity element. If $\iota_m\colon \mathbf{0} \to \mathbf{m}$ is the unique morphism, then $\iota_{m*}(1_0) = 1_m$. For $g \in G(\mathbf{n})$

$$\mu_{m,n}(1_m, g) = \mu_{m,n}(\iota_{m*}(1_0), g) = (\iota_m \sqcup \mathrm{id}_{\mathbf{n}})_* \mu_{0,n}(1_0, g) = (\iota_m \sqcup \mathrm{id}_{\mathbf{n}})_*(g)$$

by naturality. Let $g \in G(\mathbf{n})$, $h \in G(\mathbf{m})$, then

$$\begin{aligned}
\mu_{m,n}(g,h) &= \mu_{m,n}(\nu_m(1_m, g), \nu_n(h, 1_n)) = \nu_{m+n}(\mu_{m,n}(1_m, g), \mu_{m,n}(h, 1_n)) \\
&= \nu_{m+n}((\iota_m \sqcup \mathrm{id}_{\mathbf{n}})_*(g), (\mathrm{id}_{\mathbf{m}} \sqcup \iota_n)_*(h)) \\
&= \nu_{m+n}(\tau_{m,n*}(\mathrm{id}_{\mathbf{n}} \sqcup \iota_m)_*(g), \tau_{m,n*}(\iota_n \sqcup \mathrm{id}_{\mathbf{m}})_*(h)) \\
&= \tau_{m,n*}\nu_{n+m}((\mathrm{id}_{\mathbf{n}} \sqcup \iota_m)_*(g), (\iota_n \sqcup \mathrm{id}_{\mathbf{m}})_*(h)) = \tau_{m,n*}\mu_{n,m}(h, g)
\end{aligned}$$

In the last step we have used the fact that $\tau_{m,n*}$ is a group homomorphism. $\square$

Lemma 3.3.3. *Let G be an EH-$\mathcal{I}$-group with compatible inverses, let $g \in G(\mathbf{m})$, then there is a path connecting $\mu_{m,m}(g, g^{-1}) \in G(\mathbf{m}\sqcup\mathbf{m})$ and $1_{m\sqcup m} \in G(\mathbf{m}\sqcup\mathbf{m})$.*

Proof. Just as in the proof of Lemma 3.3.2 we see that

$$\mu_{m,m}(g, g^{-1}) = \nu_{m+m}((\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g), (\mathrm{id}_{\mathbf{m}} \sqcup \iota_m)_*(g^{-1}))\ .$$

But by assumption $(\mathrm{id}_{\mathbf{m}} \sqcup \iota_m)_*(g^{-1})$ is homotopic to $(\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g^{-1})$. We get

$$\nu_{m+m}((\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g), (\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g^{-1})) = (\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(\nu_m(g, g^{-1})) = (\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(1_m)$$

proving the claim. □

Let G be an EH-$\mathcal{I}$-group and let $\Delta_r \colon \mathcal{I} \to \mathcal{I}^r$ be the multidiagonal functor that maps $\mathbf{n}$ to $(\mathbf{n}, \dots, \mathbf{n}) \in \mathrm{obj}(\mathcal{I}^r)$. Then we obtain an $\mathcal{I}$-monoid $G^{(r)}$ from this via

$$G^{(r)}(\mathbf{n}) = \underbrace{(G \times \cdots \times G)}_{r \text{ times}} \circ \Delta_r(\mathbf{n}) = G(\mathbf{n})^r \ ,$$

where we define $G^{(0)}(\mathbf{n})$ to be the trivial group and $G^{(1)}(\mathbf{n}) = G(\mathbf{n})$. The multiplication is given by

$$\mu^{(r)}_{m,n} \colon G(\mathbf{m})^r \times G(\mathbf{n})^r \to (G(\mathbf{m}) \times G(\mathbf{n}))^r \to G(\mathbf{m} \sqcup \mathbf{n})^r$$

where the first map is reshuffling the factors in an order-preserving way and the second map is $\mu_{m,n} = \mu^{(1)}_{m,n}$. We can rephrase the fact that $\varphi_* \colon G(\mathbf{m}) \to G(\mathbf{n})$ is a group homomorphism for every $\varphi \in \mathrm{mor}(\mathcal{I})$ by saying that $\nu_\bullet \colon (G \times G) \circ \Delta_2 \to G$ is a natural transformation. Thus, we obtain face maps of the form

$$\begin{aligned} d_{i,n} \colon G^{(r)}(\mathbf{n}) &\to G^{(r-1)}(\mathbf{n}) \\ (g_1, \dots, g_r) &\mapsto \begin{cases} (g_2, \dots, g_r) & \text{if } i = 0 \\ (g_1, \dots, \nu_n(g_i, g_{i+1}), \dots, g_r) & \text{if } 0 < i < r \\ (g_1, \dots, g_{r-1}) & \text{if } i = r \end{cases} \end{aligned}$$

and corresponding degeneracy maps $s_{i,n} \colon G^{(r)} \to G^{(r+1)}$, which insert the identity of $G(\mathbf{n})$ after the ith element. Altogether, we see that $G^{(r)}$ is a simplicial $\mathcal{I}$-space. If we fix $\mathbf{n}$, then $G^{(\bullet)}(\mathbf{n})$ is a simplicial space. Its geometric realization is the classifying space of the group $G(\mathbf{n})$, which we will denote by $B_\nu G(\mathbf{n})$. Any morphism $\varphi \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces a simplicial map $G^{(\bullet)}(\mathbf{m}) \to G^{(\bullet)}(\mathbf{n})$ and therefore a map on the corresponding classifying spaces: $B_\nu G(\mathbf{m}) \to B_\nu G(\mathbf{n})$. This way $\mathbf{n} \mapsto B_\nu G(\mathbf{n})$ becomes an $\mathcal{I}$-space, of which we can form the homotopy colimit $(B_\nu G)_{h\mathcal{I}}$.

Alternatively, we can first form the homotopy colimit of the r-simplices to obtain the simplicial space $G^{(r)}_{h\mathcal{I}} = \mathrm{hocolim}_{\mathcal{I}}\, G^{(r)}$. Let $B_\nu(G_{h\mathcal{I}}) := |G^{(\bullet)}_{h\mathcal{I}}|$. As the notation already suggests there is not much of a difference between these two spaces.

Lemma 3.3.4. *Let $\mathcal{J}$ be a small (discrete) category, let X be a simplicial $\mathcal{J}$-space, then $c \mapsto |X^{(\bullet)}(c)|$ is a $\mathcal{J}$-space, $r \mapsto \mathrm{hocolim}_{\mathcal{J}}\, X^{(r)}$ is a simplicial space and there is a homeomorphism $|\mathrm{hocolim}_{\mathcal{J}}\, X^{(\bullet)}| \cong \mathrm{hocolim}_{\mathcal{J}} |X^{(\bullet)}|$. In particular, we have $(B_\nu G)_{h\mathcal{I}} \cong B_\nu(G_{h\mathcal{I}}) =: B_\nu G_{h\mathcal{I}}$.*

Given another category $\mathcal{J}'$, a functor $F\colon \mathcal{J} \to \mathcal{J}'$, a simplicial $\mathcal{J}'$-space X' and a natural transformation $\kappa\colon X \Rightarrow X' \circ F$, the following diagram, in which the vertical maps are induced by F and κ, commutes

$$\begin{tikzcd}
|\mathrm{hocolim}_{\mathcal{J}} X^{(\bullet)}| \arrow[r, "\cong"] \arrow[d] & \mathrm{hocolim}_{\mathcal{J}} |X^{(\bullet)}| \arrow[d] \\
|\mathrm{hocolim}_{\mathcal{J}'} X'^{(\bullet)}| \arrow[r, "\cong"'] & \mathrm{hocolim}_{\mathcal{J}'} |X'^{(\bullet)}|
\end{tikzcd}$$

Proof. The first two statements are clear, since the face and degeneracy maps are maps of $\mathcal{J}$-spaces. In particular, the space $X^{(r)}$ of r-simplices is a $\mathcal{J}$-space. Let $\mathcal{C}_X^{(r)}$ be the transport category of $X^{(r)}$ and let $N_s\mathcal{C}_X^{(r)}$ be the sth space of the nerve of $\mathcal{C}_X^{(r)}$. This is a bisimplicial space and we have

$$|\mathrm{hocolim}_{\mathcal{J}} X^{(\bullet)}| = ||N_s\mathcal{C}_X^{(r)}|_s|_r \cong ||N_s\mathcal{C}_X^{(r)}|_r|_s \cong \mathrm{hocolim}_{\mathcal{J}}|X^{\bullet}| \ .$$

The functor F in combination with the natural transformation yield a map of bisimplicial spaces $N_s\mathcal{C}_X^{(r)} \to N_s\mathcal{C}_{X'}^{(r)}$. So the last claim follows from the fact that the homeomorphism $||N_s\mathcal{C}_X^{(r)}|_s|_r \cong ||N_s\mathcal{C}_X^{(r)}|_r|_s$ is natural in the bisimplicial space. □

Lemma 3.3.5. *Let G be a stable EH-$\mathcal{I}$-group, such that $G(\mathbf{n})$ has the homotopy type of a CW-complex for each $\mathbf{n}$. The inclusion map $G(\mathbf{1})^m \to G_{h\mathcal{I}}^{(m)}$ induces a homotopy equivalence for every $m \in \mathbb{N}$. In particular, $B_\nu G(\mathbf{1}) \to B_\nu G_{h\mathcal{I}}$ is a homotopy equivalence.*

Proof. Since we assumed all maps $G(\mathbf{m}) \to G(\mathbf{n})$ to be homotopy equivalences for every $m \geq 1$, the Lemma follows from [124, Lemma 2.1] together with [129, Proposition A.1 (ii)]. □

Due to the Eckmann-Hilton condition the following diagram commutes

$$\begin{tikzcd}
G^{(r)}(\mathbf{m}) \times G^{(r)}(\mathbf{n}) \arrow[r, "\mu_{m,n}^{(r)}"] \arrow[d, "d_{i,m}\times d_{i,n}"'] & G^{(r)}(\mathbf{m} \sqcup \mathbf{n}) \arrow[d, "d_{i,m\sqcup n}"] \\
G^{(r-1)}(\mathbf{m}) \times G^{(r-1)}(\mathbf{n}) \arrow[r, "\mu_{m,n}^{(r-1)}"'] & G^{(r-1)}(\mathbf{m} \sqcup \mathbf{n})
\end{tikzcd}$$

Thus, $G^{(\bullet)}$ is a simplicial $\mathcal{I}$-monoid, which is commutative by Lemma 3.3.2. This implies that $\mathbf{n} \mapsto B_\nu G(\mathbf{n})$ is a commutative $\mathcal{I}$-monoid. Let $B_\nu G^{(s)}\colon \mathcal{I}^s \to \mathcal{T}op_*$ be the $\mathcal{I}^s$-space given by $B_\nu G^{(s)}(\mathbf{n}_1, \ldots, \mathbf{n}_s) = B_\nu G(\mathbf{n}_1) \times \cdots \times B_\nu G(\mathbf{n}_s)$. As explained in [124, section 5.2] there is a Γ-space $\Gamma(B_\nu G)$ associated to $B_\nu G$ constructed as follows: For a pointed set $(S, *)$ let $\bar{S} = S \setminus \{*\}$ and let $\mathcal{P}(\bar{S})$ be the power set of $\bar{S}$ considered as a category with respect to inclusion. We define $D(S)$ to be the category of those functors $\mathcal{P}(\bar{S}) \to \mathcal{I}$ that map disjoint unions of sets to coproducts of finite sets. The morphisms in $D(S)$ are natural transformations. Then

$$\Gamma(B_\nu G)(S) = \mathrm{hocolim}_{D(S)}\, B_\nu G^{(s)} \circ \pi_S \ ,$$

where $\pi_S \colon D(S) \to \mathcal{I}^{\bar{S}}$ is the canonical projection functor, which restricts to one-point subsets. All $G^{(r)}$ are commutative $\mathcal{I}$-monoids. Therefore we have analogous Γ-spaces $\Gamma(G^{(r)})$ and $\Gamma(G)$. Following [129, Definition 1.3], we can deloop $\Gamma(G)$: Let $Y^{(k)}(S) = \Gamma(G)([k] \wedge S)$. This is a simplicial space and $B_\mu\Gamma(G)(S) = |Y^{(\bullet)}(S)|$ is another Γ-space.

In [11] Bousfield and Friedlander discussed two model category structures on Γ-spaces: a strict and a stable one. The strict homotopy category of very special Γ-spaces is equivalent to the stable homotopy category of connective spectra [11, Theorem 5.1]. Instead of topological spaces, Bousfield and Friedlander considered (pointed) simplicial sets as a target category. The discussion in [127, Appendix B] shows that this difference is not essential. Recall from [11, Theorem 3.5] that a *strict equivalence* between Γ-spaces X and Y is a natural transformation $f \colon X \to Y$, such that $f_S \colon X(S) \to Y(S)$ is a weak equivalence for every pointed set S with $|S| \geq 2$. This agrees with the notion of strict equivalence given in [127].

Theorem 3.3.6. *Let G be a convergent EH-$\mathcal{I}$-group, such that each $G(\mathbf{n})$ has the homotopy type of a CW-complex. Let $B_\mu G_{h\mathcal{I}}$ be the classifying space of the topological monoid $(G_{h\mathcal{I}}, \mu)$. There is a strict equivalence of very special Γ-spaces $B_\mu\Gamma(G) \simeq \Gamma(B_\nu G)$ inducing a stable equivalence of the induced spectra in the stable homotopy category.*

In particular, $B_\nu G_{h\mathcal{I}} \simeq B_\mu G_{h\mathcal{I}}$. If G is stable, then $B_\mu G_{h\mathcal{I}}$ is a classifying space for principal $G(\mathbf{1})$-bundles and $B_\nu G(\mathbf{1})$ is an infinite loop space.

Proof. Let $G^{(r,s)} \colon \mathcal{I}^s \to \mathcal{T}op_*$ be the simplicial $\mathcal{I}^s$-space given by

$$G^{(r,s)}(\mathbf{n}_1, \dots, \mathbf{n}_s) = G^{(r)}(\mathbf{n}_1) \times \cdots \times G^{(r)}(\mathbf{n}_s) \ .$$

Observe that $G^{(r,s)} \circ \pi_S$ is a simplicial $D(S)$-space with $|G^{(\bullet,s)} \circ \pi_S| = B_\nu G^{(s)} \circ \pi_S$, therefore Lemma 3.3.4 yields a homeomorphism $\Gamma(B_\nu G)(S) \cong |\Gamma(G^{(\bullet)})(S)|$. By the commutativity of the diagram in Lemma 3.3.4, this is natural in S. Let $|\Gamma(G^{(\bullet)})|(S) := |\Gamma(G^{(\bullet)})(S)|$. We obtain a levelwise homeomorphism of Γ-spaces $\Gamma(B_\nu G) \cong |\Gamma(G^{(\bullet)})|$. In particular, this is a strict equivalence. The projection maps $\pi_k \colon G^{(r)}(\mathbf{n}) = G(\mathbf{n})^r \to G(\mathbf{n}) = G^{(1)}(\mathbf{n})$ induce

$$\pi \colon \Gamma(G^{(r)})(S) \to \Gamma(G)(S) \times \cdots \times \Gamma(G)(S) \ .$$

For $S = S^0$ we have $\Gamma(G^{(r)})(S) = G^{(r)}_{h\mathcal{I}}$, $\Gamma(G)(S) = G_{h\mathcal{I}}$. π fits into the commutative diagram

$$\begin{tikzcd} G^{(r)}_{h\mathcal{I}} \arrow[r, "\pi"] & G_{h\mathcal{I}} \times \cdots \times G_{h\mathcal{I}} \\ \mathrm{Tel}(G^{(r)}) \arrow[u, "\simeq"] \arrow[r, "\widehat{\pi}"] & \mathrm{Tel}(G) \times \cdots \times \mathrm{Tel}(G) \arrow[u, "\simeq"'] \end{tikzcd}$$

in which the homotopy equivalences follow from [124, Lemma 2.1], since G is convergent. Note that $\pi_k(\mathrm{Tel}(G)^{(r)}) = \varinjlim(\pi_k(G(\mathbf{n})) \times \cdots \times \pi_k(G(\mathbf{n})))$, where the limit runs over the directed poset $\mathbb{N}_0$. Moreover, $\pi_k(\mathrm{Tel}(G)^r) = \varinjlim(\pi_k(G(\mathbf{n}_1)) \times \cdots \times \pi_k(G(\mathbf{n}_r)))$, where the limit runs over the poset $\mathbb{N}_0^r$. The map $\widehat{\pi}$ is induced by the diagonal $\Delta_r \colon \mathbb{N}_0 \to \mathbb{N}_0^r$. Since the subset $\Delta_r(\mathbb{N}_0)$ is cofinal in $\mathbb{N}_0^r$, we obtain an isomorphism on all homotopy groups and therefore – by our assumption on G – a homotopy equivalence.

Let now S be arbitrary, let $s = |S|$. Let $G^{(r,s)}_{h\mathcal{I}} = \mathrm{hocolim}_{\mathcal{I}^s} G^{(r,s)}$. By [124, Lemma 5.1], the natural map $\Gamma(G^{(r)})(S) \to G^{(r,s)}_{h\mathcal{I}}$ is an equivalence. From the above, we obtain that $\widetilde{\pi} \colon G^{(r,s)}_{h\mathcal{I}} \to G^{(1,s)}_{h\mathcal{I}} \times \cdots \times G^{(1,s)}_{h\mathcal{I}}$ is an equivalence. The diagram

$$\begin{tikzcd} \Gamma(G^{(r)})(S) \arrow[r, "\pi"] \arrow[d, "\simeq"'] & \Gamma(G)(S) \times \cdots \times \Gamma(G)(S) \arrow[d, "\simeq"] \\ G^{(r,s)}_{h\mathcal{I}} \arrow[r, "\widetilde{\pi}", "\simeq"'] & G^{(1,s)}_{h\mathcal{I}} \times \cdots \times G^{(1,s)}_{h\mathcal{I}} \end{tikzcd}$$

shows that π is an equivalence for arbitrary S, which in turn implies that the induced map $\pi \colon B_\mu\Gamma(G^{(r)})(S) \to B_\mu\Gamma(G)(S)\times\cdots\times B_\mu\Gamma(G)(S)$ is an equivalence as well. Now observe that $B_\mu|\Gamma(G^{(\bullet)})|(S) \cong |B_\mu\Gamma(G^{(\bullet)})|(S)$ and $[k] \mapsto B_\mu\Gamma(G^{(k)})(S)$ is a simplicial space satisfying the properties of [129, Proposition 1.5]. In particular, $\pi_0(B_\mu\Gamma(G)(S)) \cong \pi_0((B_\mu G_{h\mathcal{I}})^s)$ is trivial. Therefore $B_\mu\Gamma(G)(S) \to \Omega|B_\mu\Gamma(G^{(\bullet)})|(S) \cong \Omega B_\mu|\Gamma(G^{(\bullet)})|(S)$ is a homotopy equivalence, which is natural in S. Altogether we obtain a sequence of strict equivalences

$$\Gamma(B_\nu G) \simeq \Omega B_\mu\Gamma(B_\nu G) \simeq \Omega B_\mu|\Gamma(G^{(\bullet)})| \simeq B_\mu\Gamma(G)\ .$$

We used that $B_\nu G$ is connected (and therefore $\Gamma(B_\nu G)$ is a very special Γ-space) in the first step. $S = S^0$ together with Lemma 3.3.5 yields $B_\nu G(\mathbf{1}) \simeq B_\nu G_{h\mathcal{I}} \simeq B_\mu G_{h\mathcal{I}}$ in case G is stable. $\square$

3.3.1 Actions of Eckmann-Hilton $\mathcal{I}$-groups on spectra

As was alluded to in the introduction, the compatibility diagram of an EH-$\mathcal{I}$-group G from Definition 3.3.1 enables us to talk about the action of G on a commutative symmetric ring spectrum R.

Definition 3.3.7. Let G be an EH-$\mathcal{I}$-group and let R be a commutative symmetric ring spectrum. Then G is said to *act on* R, if $G(\mathbf{n})$ acts on R_n via $\kappa_n \colon G(\mathbf{n}) \times R_n \to R_n$ such that

(i) κ_n preserves the basepoint of R_n and is Σ_n-equivariant, where Σ_n acts on $G(\mathbf{n}) \times R_n$ diagonally and $G(\mathbf{0})$ acts trivially on R_0,

(ii) the action is compatible with the multiplication in the sense that the following diagram commutes

$$\begin{tikzcd} G(\mathbf{m}) \times R_m \times G(\mathbf{n}) \times R_n \arrow[r, "\kappa_m \times \kappa_n"] \arrow[d, "(\mu^G_{m,n} \times \mu^R_{m,n}) \circ \tau"'] & R_m \wedge R_n \arrow[d, "\mu^R_{m,n}"] \\ G(\mathbf{m} \sqcup \mathbf{n}) \times R_{m+n} \arrow[r, "\kappa_{m+n}"'] & R_{m+n} \end{tikzcd}$$

where τ denotes the map that switches the two inner factors.

(iii) the action is compatible with stabilization in the sense that the following diagram commutes for $l, m, n \in \mathbb{N}_0$ with $l + m = n$, $\sigma_{l,m} = \mu_{l,m} \circ (\eta_l \wedge \mathrm{id}_{R_m})$ and the order preserving inclusion $\iota_{m,n} \colon \mathbf{m} \to \mathbf{n}$ onto the last m elements

$$\begin{tikzcd} G(\mathbf{m}) \times (S^l \wedge R_m) \arrow[r, "\mathrm{id}_{S^l} \wedge \kappa_m"] \arrow[d, "\iota_{m,n*} \times \sigma_{l,m}"'] & S^l \wedge R_m \arrow[d, "\sigma_{l,m}"] \\ G(\mathbf{n}) \times R_n \arrow[r, "\kappa_n"'] & R_n \end{tikzcd}$$

Theorem 3.3.8. *An action of an EH-$\mathcal{I}$-group G on a commutative symmetric ring spectrum R defines a map of $\mathcal{I}$-monoids $G \to \Omega^\infty(R)$, which sends $g \in G(\mathbf{n})$ to $g \cdot \eta_n := \kappa_n(g, \eta_n) \in \Omega^\infty(R)(\mathbf{n}) = \Omega^n R_n$. If G has compatible inverses, this factors over a morphism $G \to \Omega^\infty(R)^*$ of commutative $\mathcal{I}$-monoids, which deloops to a map $B_\mu\Gamma(G) \to B_\mu\Gamma(\Omega^\infty(R)^*)$. In particular, we obtain $B_\mu G_{h\mathcal{I}} \to BGL_1(R)$.*

Proof. To see that $G \to \Omega^\infty(R)$ really defines a natural transformation, observe that each morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ factors as $\alpha = \bar{\alpha} \circ \iota_{m,n}$ with $\bar{\alpha} \in \Sigma_n$ as explained in the paragraph after Definition 3.2.2. Now note that

$$\begin{aligned} \alpha_*(g \cdot \eta_m) &= \bar{\alpha} \circ \mu^R_{l,m}(\eta_l \wedge (g \cdot \eta_m)) \circ \bar{\alpha}^{-1} = \bar{\alpha} \circ \iota_{m,n*}(g) \cdot \mu^R_{l,m}(\eta_l \wedge \eta_m) \circ \bar{\alpha}^{-1} \\ &= \alpha_*(g) \cdot \bar{\alpha} \circ \eta_n \circ \bar{\alpha}^{-1} = \alpha_*(g) \cdot \eta_n \end{aligned}$$

where we used (i) and (iii) of Definition 3.3.7. That $G \to \Omega^\infty(R)$ is a morphism of $\mathcal{I}$-monoids is a consequence of (ii). Indeed, for $g \in G(\mathbf{m})$, $h \in G(\mathbf{n})$

$$\mu^R_{m,n}(g \cdot \eta_m, h \cdot \eta_n) = \mu^G_{m,n}(g,h) \cdot \mu^R_{m,n}(\eta_m, \eta_n) = \mu^G_{m,n}(g,h) \cdot \eta_{m+n} \ .$$

If G has compatible inverses, then g^{-1} provides a stable inverse of $g \in G(\mathbf{n})$ by Lemma 3.3.3. Forming homotopy colimits, we obtain a morphism of topological monoids $G_{h\mathcal{I}} \to GL_1(R)$, which deloops. $\square$

3.3.2 Eckmann-Hilton $\mathcal{I}$-groups and permutative categories

Given a permutative category $(\mathcal{C}, \otimes)$ and an object $x \in \mathrm{obj}(\mathcal{C})$, there is a canonical commutative $\mathcal{I}$-monoid E_x associated to it: Let $E_x(\mathbf{n}) = \mathrm{End}_{\mathcal{C}}(x^{\otimes n})$ (with $x^{\otimes 0} = 1_{\mathcal{C}}$), let $\bar{\alpha} \in \Sigma_n$ be the permutation associated to a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ as above and define $\alpha_*\colon E_x(\mathbf{m}) \to E_x(\mathbf{n})$ by sending an endomorphism f to $\bar{\alpha} \circ (\mathrm{id}_{x^{\otimes n-m}} \otimes f) \circ \bar{\alpha}^{-1}$, where the permutation group Σ_n acts on $x^{\otimes n}$ using the symmetry of $\mathcal{C}$. The monoid structure of E_x is given by

$$\mu_E\colon E_x(\mathbf{m}) \times E_x(\mathbf{n}) \to E_x(\mathbf{m} \sqcup \mathbf{n}) \quad ; \quad (f,g) \mapsto f \otimes g \ .$$

Let $A_x(\mathbf{n}) = \mathrm{Aut}_{\mathcal{C}}(x^{\otimes n})$ together with the analogous structures as described above. This is an Eckmann-Hilton $\mathcal{I}$-group. Let $\mathcal{C}_x$ be the full permutative subcategory of $\mathcal{C}$ containing the objects $x^{\otimes n}$, $n \in \mathbb{N}_0$.

Definition 3.3.9. A strict symmetric monoidal functor $\theta\colon \mathcal{I} \to \mathcal{C}_x$ will be called a *stabilization of* x, if $\theta(\mathbf{1}) = x$ and for each morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ and each $f \in E_x(\mathbf{m})$ the following diagram commutes:

$$\begin{tikzcd}
\theta(\mathbf{m}) \arrow[r, "f"] \arrow[d, "\theta(\alpha)"'] & \theta(\mathbf{m}) \arrow[d, "\theta(\alpha)"] \\
\theta(\mathbf{n}) \arrow[r, "\alpha_*(f)"'] & \theta(\mathbf{n})
\end{tikzcd}$$

Lemma 3.3.10. *Let $\iota_\mathbf{1}\colon \mathbf{0} \to \mathbf{1}$ be the unique morphism in $\mathcal{I}$. The map that associates to a stabilization θ the morphism $\theta(\iota_\mathbf{1}) \in \mathrm{hom}_{\mathcal{C}}(1_{\mathcal{C}}, x)$ yields a bijection between stabilizations and elements in $\mathrm{hom}_{\mathcal{C}}(1_{\mathcal{C}}, x)$.*

Proof. Note that a stabilization is completely fixed by knowing $\theta(\iota_\mathbf{1})$, therefore the map is injective. Let $\varphi \in \mathrm{hom}_{\mathcal{C}}(1_{\mathcal{C}}, x)$. Define $\theta(\mathbf{n}) = x^{\otimes n}$ on objects. Let $\alpha\colon \mathbf{m} \to \mathbf{n}$ be a morphism in $\mathcal{I}$ and let $\alpha = \bar{\alpha} \circ \iota_{m,n}$ be the factorization as explained after Definition 3.2.2. Define $\theta(\iota_{m,n}) = \varphi^{\otimes(n-m)} \otimes \mathrm{id}_{x^{\otimes m}}\colon x^{\otimes m} = 1_{\mathcal{C}}^{\otimes(n-m)} \otimes x^{\otimes m} \to x^{\otimes n}$ and let $\theta(\bar{\alpha})$ be the permutation of the tensor factors. The decomposition $\alpha = \bar{\alpha} \circ \iota_{m,n}$ is *not* functorial, due to the fact that

$\overline{\beta \circ \alpha}$ and $\bar{\beta} \circ (\mathrm{id} \sqcup \bar{\alpha})$ differ by a permutation. Nevertheless $\theta(\alpha) = \theta(\bar{\alpha}) \circ \theta(\iota_{m,n})$ turns out to be functorial due to the permutation invariance of $\varphi^{\otimes k}$. It is straightforward to check that this is also strict symmetric monoidal. Let $f \colon x^{\otimes m} \to x^{\otimes m}$ be in $E_x(\mathbf{m})$. We have

$$\theta(\alpha) \circ f = \bar{\alpha} \circ (\varphi^{\otimes(n-m)} \otimes \mathrm{id}_{x^{\otimes m}}) \circ f = \bar{\alpha} \circ (\mathrm{id}_{x^{\otimes n-m}} \otimes f) \circ \bar{\alpha}^{-1} \circ \theta(\alpha) = \alpha_*(f) \circ \theta(\alpha) \ .$$

This shows that the map is also surjective. □

As sketched in [129] there is a Γ-space $\Gamma(\mathcal{C})$ associated to a Γ-category $\mathcal{A}_{\mathcal{C}}$, which is constructed as follows: Let S be a finite pointed set and denote by $\bar{S}$ the complement of the basepoint of S, then the objects of $\mathcal{A}_{\mathcal{C}}(S)$ are families $\{x_U \in \mathrm{obj}(\mathcal{C}) \mid U \subset \bar{S}\}$ together with isomorphisms $\alpha_{U,V} \colon x_U \otimes x_V \to x_{U \cup V}$, whenever $U \cap V = \emptyset$, compatible with the symmetry of $\mathcal{C}$ and such that $x_\emptyset = 1_{\mathcal{C}}$ and $x_U \otimes x_\emptyset \to x_U$ is the identity. The morphisms of $\mathcal{A}_{\mathcal{C}}(S)$ are families of morphisms $\beta_{U,U'} \colon x_U \to x_{U'}$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
x_U \otimes x_V \arrow[r, "\alpha_{U,V}"] \arrow[d, "\beta_{U,U'} \otimes \beta_{V,V'}"'] & x_{U \cup V} \arrow[d, "\beta_{U \cup V, U' \cup V'}"] \\
x_{U'} \otimes x_{V'} \arrow[r, "\alpha_{U',V'}"'] & x_{U' \cup V'}
\end{tikzcd}$$

commutes. We define $\Gamma(\mathcal{C})(S) = |\mathcal{A}_{\mathcal{C}}(S)|$. Let $\mathcal{C}_x$ be the full (permutative) subcategory of $\mathcal{C}$ containing the objects $x^{\otimes n}$ for $n \in \mathbb{N}_0$.

In the next lemma we will introduce a technical tool from simplicial homotopy theory: The nerve of a topological category $\mathcal{D}$ is the simplicial space $N_n\mathcal{D} = \mathrm{Fun}([n], \mathcal{D})$, where $[n]$ is the category of the directed poset $\{0, \dots, n\}$. We define the *double nerve* of $\mathcal{D}$ as the *bi*simplicial space $N_\bullet N_\bullet \mathcal{D}$ with $N_m N_n \mathcal{D} = \mathrm{Fun}([m] \times [n], \mathcal{D})$. It consists of an $m \times n$-array of commuting squares in $\mathcal{D}$. The diagonal functors $\mathrm{diag}_n \colon [n] \to [n] \times [n]$ induce a simplicial map $\varphi_n \colon N_n N_n \mathcal{D} \to N_n \mathcal{D}$.

Lemma 3.3.11. *The simplicial map $\varphi_\bullet$ induces a homotopy equivalence $|N_\bullet N_\bullet \mathcal{D}| \to |N_\bullet \mathcal{D}|$.*

Proof. To construct the homotopy inverse, let $\max_n \colon [n] \times [n] \to [n]$ be given by $\max_n(k, \ell) = \max\{k, \ell\}$, which completely determines its value on morphisms. We have $\max_n \circ \mathrm{diag}_n = \mathrm{id}_{[n]}$. Let $(k, \ell) \in [n] \times [n]$. There is a unique morphism $\kappa_{(k,\ell)} \colon (k, \ell) \to (\max\{k, \ell\}, \max\{k, \ell\})$. Therefore there is a natural transformation $\kappa \colon \mathrm{id}_{[n] \times [n]} \to \mathrm{diag}_n \circ \max_n$. From κ, we can construct a functor $h \colon [n] \times [n] \times [1] \to [n] \times [n]$, which induces

$$H \colon N_n N_n \mathcal{D} \times \hom([n] \times [n], [1]) \to N_n N_n \mathcal{D} \ ; \ (F, f) \mapsto F \circ h \circ (\mathrm{id}_{[n] \times [n]} \times f) \circ \mathrm{diag}_{[n] \times [n]} \ .$$

After geometric realization, H yields a homotopy inverse of $\mathrm{diag}_n \circ \max_n$. □

Let $S_\otimes\colon \mathcal{C}_x \to \mathcal{C}_x$ be the functor $x^{\otimes n} \mapsto x^{\otimes(n+1)}$ and $f \mapsto f \otimes \mathrm{id}_x$. Observe that

$$|\mathrm{Tel}(N_k\mathcal{C}_x; N_kS_\otimes)|_k \cong \mathrm{Tel}(|N_\bullet\mathcal{C}_x|; |N_\bullet S_\otimes|) = \mathrm{Tel}(|N_\bullet\mathcal{C}_x|)$$

by Lemma 3.3.4 and that θ provides a natural transformation $\mathrm{id}_{\mathcal{C}_x} \to S_\otimes$. Therefore we obtain a map $\mathrm{Tel}(|N_\bullet\mathcal{C}_x|) \to |N_\bullet N_\bullet\mathcal{C}_x|$.

Lemma 3.3.12. *The map* $\mathrm{Tel}(|N_\bullet\mathcal{C}_x|) \to |N_\bullet N_\bullet\mathcal{C}_x|$ *constructed in the last paragraph is a homotopy equivalence.*

Proof. θ yields a natural transformation $\mathrm{id}_{\mathcal{C}_x} \to S_\otimes$. Therefore $|N_\bullet S_\otimes|$ is homotopic to the identity and the map $|N_\bullet\mathcal{C}_x| \to \mathrm{Tel}(|N_\bullet\mathcal{C}_x|)$ onto the 0-skeleton is a homotopy equivalence. Likewise, $|N_\bullet\mathcal{C}_x| \to |N_\bullet N_\bullet\mathcal{C}_x|$ induced by the simplicial map $N_\ell\,\mathcal{C}_x \to N_\ell N_\ell\,\mathcal{C}_x$ sending a diagram to the corresponding square that has ℓ copies of the diagram in its rows and only identities as vertical maps is a homotopy equivalence by Lemma 3.3.11 since composition with $\varphi_\bullet$ yields the identity on $N_\ell\,\mathcal{C}_x$. The statement now follows from the commutative triangle

$$\begin{tikzcd}
\mathrm{Tel}(|N_\bullet\mathcal{C}_x|) \arrow[rr] & & |N_\bullet N_\bullet\mathcal{C}_x| \\
& |N_\bullet\mathcal{C}_x| \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

which finishes the proof. □

Theorem 3.3.13. *Let $x \in \mathrm{obj}(\mathcal{C})$ and let $\theta\colon \mathcal{I} \to \mathcal{C}_x$ be a stabilization of x. Let $G = A_x$. There is a map of Γ-spaces*

$$\Phi_\theta\colon \Gamma(B_\nu G) \to \Gamma(\mathcal{C}_x)\ .$$

If G is convergent and there exists an unbounded non-decreasing sequence of natural numbers λ_n, such that $B_\nu G(\mathbf{m}) \to |N_\bullet\mathcal{C}_x|$ is λ_n-connected for all $m > n$, then Φ_θ is an equivalence as well.

Proof. Lemma 3.3.4 yields a homeomorphism $\Gamma(B_\nu G)(S) \cong |\Gamma(G^{(\bullet)})(S)|$ with

$$\Gamma(G^{(r)})(S) = \mathrm{hocolim}_{D(S)}\, G^{(r,s)} \circ \pi_S\ .$$

The last space is the geometric realization of the transport category $\mathcal{T}_G^{(r)}(S)$ with object space $\coprod_{d\in\mathrm{obj}(D(S))} G^{(r,s)}(\pi_S(d))$ and morphism space

$$\mathrm{mor}(\mathcal{T}_G^{(r)}(S)) = \coprod_{d,d'\in\mathrm{obj}(D(S))} G^{(r,s)}(\pi_S(d)) \times \mathrm{hom}_{D(S)}(d,d')\ .$$

Given a diagram $d \in \operatorname{obj}(D(S))$, we obtain $\mathbf{n}_U = d(U) \in \operatorname{obj}(\mathcal{I})$ for every $U \subset \bar{S}$ and define $\mathbf{n}_i = d(\{i\})$. Let $(g_1, \dots g_s) \in G^{(r)}(\mathbf{n}_1) \times \cdots \times G^{(r)}(\mathbf{n}_s)$. We may interpret g_i as an r-tupel $(g_{j,i})_{j\in\{1,\dots,r\}}$ of automorphisms $g_{j,i} \in G(\mathbf{n}_i) = \operatorname{Aut}_{\mathcal{C}}(\theta(\mathbf{n}_i))$. If $U, V \subset \bar{S}$ are two subsets with $U \cap V = \emptyset$, then d yields two morphisms $\mathbf{n}_U \to \mathbf{n}_{U\cup V}$ and $\mathbf{n}_V \to \mathbf{n}_{U\cup V}$, which form an isomorphism $\iota_{U,V} \colon \mathbf{n}_U \sqcup \mathbf{n}_V \to \mathbf{n}_{U\cup V}$ in $\mathcal{I}$. Let $\alpha_{U,V} = \theta(\iota_{U,V}) \colon \theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) \to \theta(\mathbf{n}_{U\cup V})$. Let $\iota_U^i \colon \mathbf{n}_i \to \mathbf{n}_U$ be induced by the inclusion $\{i\} \subset U$ and define $g_{j,U} = \prod_{i\in U} \iota^i_{U*}(g_{j,i})$. This does not depend on the order, in which the factors are multiplied. The elements $g_{j,U}$ fit into a commutative diagram

$$\begin{tikzcd}
\theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) \arrow[r, "\alpha_{U,V}"] \arrow[d, "g_{j,U}\otimes g_{j,V}"'] & \theta(\mathbf{n}_{U\cup V}) \arrow[d, "g_{j,U\cup V}"] \\
\theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) \arrow[r, "\alpha_{U,V}"'] & \theta(\mathbf{n}_{U\cup V})
\end{tikzcd}$$

Thus, we can interpret the families $g_{j,U}$ for $j \in \{1, \dots, r\}$ as an r-chain of automorphisms of the object $(\theta(\mathbf{n}_U), \alpha_{U,V})$ in $\mathcal{A}_{\mathcal{C}_x}(S)$.

Let d' be another diagram, let $\mathbf{m}_U = d'(U)$, $\iota'_{U,V} \colon \mathbf{m}_U \sqcup \mathbf{m}_V \to \mathbf{m}_{U\cup V}$ and let $\beta_{U,V} = \theta(\iota'_{U,V})$. A natural transformation $d \to d'$ consists of morphisms $\varphi_U \colon \mathbf{n}_U \to \mathbf{m}_U$ for every $U \in \mathcal{P}(\bar{S})$. Let $f_U = \theta(\varphi_U)$, then the following diagrams commute:

$$\begin{tikzcd}
\theta(\mathbf{n}_U) \arrow[r, "g_{j,U}"] \arrow[d, "f_U"'] & \theta(\mathbf{n}_U) \arrow[d, "f_U"] \\
\theta(\mathbf{m}_U) \arrow[r, "\varphi_{U*}(g_{j,U})"'] & \theta(\mathbf{m}_U)
\end{tikzcd}
\qquad
\begin{tikzcd}
\theta(\mathbf{n}_U) \otimes \theta(\mathbf{n}_V) \arrow[r, "\alpha_{U,V}"] \arrow[d, "f_U\otimes f_V"'] & \theta(\mathbf{n}_{U\cup V}) \arrow[d, "f_{U\cup V}"] \\
\theta(\mathbf{m}_U) \otimes \theta(\mathbf{m}_V) \arrow[r, "\beta_{U,V}"'] & \theta(\mathbf{m}_{U\cup V})
\end{tikzcd}$$

the first since θ is a stabilization, the second by naturality of the transformation. This is compatible with respect to composition of natural transformations of diagrams. Altogether we have constructed a bisimplicial map $N_\ell \mathcal{T}_G^{(r)}(S) \to N_\ell N_r \mathcal{A}_{\mathcal{C}_x}(S)$. Combining this with $N_\ell N_\ell \mathcal{A}_{\mathcal{C}_x}(S) \to N_\ell \mathcal{A}_{\mathcal{C}_x}(S)$ from Lemma 3.3.11, we obtain $\Gamma(B_\nu G)(S) \to \Gamma(\mathcal{C}_x)(S)$ after geometric realization.

It remains to be proven that this map is functorial with respect to morphisms $\kappa \colon S \to T$ in Γ^{op}. Recall that

$$N_\ell \mathcal{T}_G^{(r)}(S) = \coprod_{d_1,\dots,d_\ell \in \operatorname{obj}(D(S))} \prod_{s\in S} G^{(r)}(d_1(\{s\})) \times \hom_{D(S)}(d_1, d_2) \times \cdots \times \hom_{D(S)}(d_{\ell-1}, d_\ell)$$

Let $\kappa_* \colon N_\ell \mathcal{T}_G^{(r)}(S) \to N_\ell \mathcal{T}_G^{(r)}(T)$ be the induced map as defined in [124, section 5.2]. We have

$$\kappa_* \left((g_{j,i})_{i\in S, j\in\{0,\dots,r\}}, \varphi^1, \dots, \varphi^{\ell-1} \right) = \left((h_{j,k})_{k\in T, j\in\{0,\dots,r\}}, \kappa_*\varphi^1, \dots, \kappa_*\varphi^{\ell-1} \right)$$

where $h_{j,k} = \prod_{i\in\kappa^{-1}(k)} \iota^i_{\kappa^{-1}(k)*}(g_{j,i})$. If the left hand side lies in the component $(d_1, \dots, d_\ell)$, then the right hand side is in $(\kappa_* d_1, \dots, \kappa_* d_\ell)$ with $(\kappa_* d_m)(V) = d_m(\kappa^{-1}(V))$ for $V \subset \bar{T}$. Likewise $(\kappa_*\varphi^m)_V = \varphi^m_{\kappa^{-1}(V)}$. The functor $\kappa_* \colon \mathcal{A}_{\mathcal{C}_x}(S) \to \mathcal{A}_{\mathcal{C}_x}(T)$ sends the object $(x_U, \alpha_{U,V})_{U,V\subset\bar{S}}$ to $(x_{\kappa^{-1}(\tilde{U})}, \alpha_{\kappa^{-1}(\tilde{U}),\kappa^{-1}(\tilde{V})})_{\tilde{U},\tilde{V}\subset\bar{T}}$ and is defined analogously on morphisms. Observe that the composition $N_\ell\mathcal{T}^{(r)}_G(S) \to N_\ell\mathcal{T}^{(r)}_G(T) \to N_\ell N_r\mathcal{A}_{\mathcal{C}_x}(T)$ maps $(g_{j,i})_{i\in S, j\in\{0,\dots,r\}}$ to the r-chain of automorphisms given by $h_{j,V} \colon \theta(\kappa_* d(V)) \to \theta(\kappa_* d(V))$ with

$$h_{j,V} = \prod_{k\in V}\prod_{i\in\kappa^{-1}(k)} (\kappa_* d)(\iota^k_V) \circ d(\iota^i_{\kappa^{-1}(k)})(g_{j,i}) = \prod_{i\in\kappa^{-1}(V)} d(\iota^i_{\kappa^{-1}(V)})(g_{j,i}) = g_{j,\kappa^{-1}(V)}$$

for $V \subset \bar{T}$. The transformations φ^m are mapped to $\theta(\varphi^m_{\kappa^{-1}(V)}) = f^m_{\kappa^{-1}(V)}$. This implies the commutativity of

$$\begin{tikzcd}
N_\ell\mathcal{T}^{(r)}_G(S) \arrow[r, "\kappa_*"] \arrow[d, "\Phi_\theta"'] & N_\ell\mathcal{T}^{(r)}_G(T) \arrow[d, "\Phi_\theta"] \\
N_\ell N_r\mathcal{A}_{\mathcal{C}_x}(S) \arrow[r, "\kappa_*"] & N_\ell N_r\mathcal{A}_{\mathcal{C}_x}(T)
\end{tikzcd}$$

and therefore functoriality after geometric realization.

To see that Φ_θ is an equivalence for each S it suffices to check that $B_\nu G_{h\mathcal{I}} \to |N_\bullet\mathcal{C}_x|$ is a homotopy equivalence due to the following commutative diagram

$$\begin{tikzcd}
\Gamma(B_\nu G)(S) \arrow[r] \arrow[d, "\simeq"'] & \Gamma(\mathcal{C}_x)(S) \arrow[d, "\simeq"] \\
(B_\nu G_{h\mathcal{I}})^s \arrow[r] & |N_\bullet\mathcal{C}_x|^s
\end{tikzcd}$$

Let $\mathrm{Tel}(|N_\bullet\mathcal{C}_x|)$ be the telescope of $|N_\bullet S_\otimes|$ as defined above. The conditions on the maps $B_\nu G(\mathbf{n}) \to |N_\bullet\mathcal{C}_x|$ ensure that $\mathrm{Tel}(B_\nu G) \to \mathrm{Tel}(|N_\bullet\mathcal{C}_x|)$ is a homotopy equivalence. Now we have the following commutative diagram

$$\begin{tikzcd}
B_\nu G_{h\mathcal{I}} \arrow[r] & |N_\bullet N_\bullet\mathcal{C}_x| \arrow[r, "\simeq"] & |N_\bullet\mathcal{C}_x| \\
\mathrm{Tel}(B_\nu G) \arrow[r, "\simeq"] \arrow[u, "\simeq"] & \mathrm{Tel}(|N_\bullet\mathcal{C}_x|) \arrow[u, "\simeq"'] &
\end{tikzcd}$$

in which the upper horizontal map is the one we are looking for. This finishes the proof. $\square$

3.4 Strongly self-absorbing C^*-algebras and $gl_1(KU^A)$

A C^*-algebra A is called *strongly self-absorbing* if it is separable, unital and there exists a $*$-isomorphism $\psi\colon A \to A \otimes A$ such that ψ is approximately unitarily equivalent to the map $l\colon A \to A \otimes A$, $l(a) = a \otimes 1_A$ [139]. This means that there is a sequence of unitaries (u_n) in A such that $\|u_n\varphi(a)u_n^* - l(a)\| \to 0$ as $n \to \infty$ for all $a \in A$. In fact, it is a consequence of [51, Theorem 2.2] and [150] that ψ, l and $r\colon A \to A \otimes A$ with $r(a) = 1_A \otimes a$ are homotopy equivalent and in fact the group $\mathrm{Aut}(A)$ is contractible [47]. The inverse isomorphism ψ^{-1} equips $K_*(C(X) \otimes A)$ with a ring structure induced by the tensor product. By homotopy invariance of K-theory, the K_0-class of the constant map on X with value $1 \otimes e$ for a rank 1-projection $e \in \mathbb{K}$ is the unit of this ring structure. Given a separable, unital, strongly self-absorbing C^*-algebra A, the functor $X \mapsto K_*(C(X) \otimes A)$ is a multiplicative cohomology theory on finite CW-complexes with respect to this ring structure.

3.4.1 A commutative symmetric ring spectrum representing K-theory

A C^*-algebra B is *graded*, if it comes equipped $\mathbb{Z}/2\mathbb{Z}$-action, i.e. a $*$-automorphism $\alpha\colon B \to B$, such that $\alpha^2 = \mathrm{id}_B$. A *graded homomorphism* $\varphi\colon (B, \alpha) \to (B', \alpha')$ has to satisfy $\varphi \circ \alpha = \alpha' \circ \varphi$. The algebraic tensor product $B \mathbin{\widehat{\odot}} B'$ can be equipped with the multiplication and $*$-operation

$$(a \mathbin{\widehat{\odot}} b)(a' \mathbin{\widehat{\odot}} b') = (-1)^{\partial b \cdot \partial a'}(aa' \mathbin{\widehat{\odot}} bb') \quad \text{and} \quad (a \mathbin{\widehat{\odot}} b)^* = (-1)^{\partial a \cdot \partial b}(a^* \mathbin{\widehat{\odot}} b^*)$$

where $a, a' \in B$ and $b, b' \in B'$ are homogeneous elements and ∂a denotes the degree of a. It is graded via $\partial(a \mathbin{\widehat{\odot}} b) = \partial a + \partial b$ modulo 2. The (minimal) graded tensor product $B \mathbin{\widehat{\otimes}} B'$ is the completion of $B \mathbin{\widehat{\odot}} B'$ with respect to the tensor product of faithful representations of B and B' on graded Hilbert spaces. For details we refer the reader to [18, section 14.4].

We define $\widehat{S} = C_0(\mathbb{R})$ with the grading by even and odd functions. The Clifford algebra $\mathbb{C}\ell_1$ will be spanned by the even element 1 and the odd element c with $c^2 = 1$. The algebra $\widehat{\mathbb{K}}$ will denote the graded compact operators on a graded Hilbert space $H = H_0 \oplus H_1$ with grading Ad_u, $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, whereas we will use $\mathbb{K}$ for the *trivially* graded compact operators. If we take tensor products between a graded C^*-algebra and a trivially graded one, e.g. $\mathbb{C}\ell_1 \widehat{\otimes} \mathbb{K}$, we will write $\otimes$ instead of $\widehat{\otimes}$. It is a consequence of [18, Corollary 14.5.5] that $(\mathbb{C}\ell_1 \otimes \mathbb{K}) \mathbin{\widehat{\otimes}} (\mathbb{C}\ell_1 \otimes \mathbb{K}) \cong M_2(\mathbb{C}) \otimes \mathbb{K} \cong \widehat{\mathbb{K}}$, where $M_2(\mathbb{C})$ is graded by Ad_u with $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

As is explained in [53, section 3.2], [79, section 4], the algebra $\widehat{S}$ carries a counital, coassociative and cocommutative coalgebra structure. This arises from a $1:1$-correspondence between essential graded $*$-homomorphisms $\widehat{S} \to A$ and odd, self-adjoint, regular unbounded multipliers of A [141, Proposition 3.1]. Let X be the multiplier corresponding to the identity map on $\widehat{S}$, then the comultiplication $\Delta\colon \widehat{S} \to \widehat{S} \otimes \widehat{S}$ is given by $1 \mathbin{\widehat{\otimes}} X + X \mathbin{\widehat{\otimes}} 1$, whereas the counit $\epsilon\colon \widehat{S} \to \mathbb{C}$ corresponds to $0 \in \mathbb{C}$, i.e. it maps $f \mapsto f(0)$.

Definition 3.4.1. Let A be a separable, unital, strongly self-absorbing and trivially graded C^*-algebra. Let $KU^A_\bullet$ be the following sequence of spaces

$$KU^A_n = \hom_{\mathrm{gr}}(\widehat{S}, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n}) \ ,$$

where the graded homomorphisms are equipped with the point-norm topology.

KU^A_n is pointed by the 0-homomorphism and carries a basepoint preserving Σ_n-action by permuting the factors of the graded tensor product (this involves signs!). We set $B^{\widehat{\otimes} 0} = \mathbb{C}$ and observe that KU^A_0 is the two-point space consisting of the 0-homomorphism and the evaluation at 0, which is the counit of the coalgebra structure on $\widehat{S}$.

Let $\mu_{m,n}$ be the following family of maps

$$\mu_{m,n}\colon KU^A_m \wedge KU^A_n \to KU^A_{m+n} \quad ; \quad \varphi \wedge \psi \mapsto (\varphi \mathbin{\widehat{\otimes}} \psi) \circ \Delta$$

To construct the maps $\eta_n\colon S^n \to KU^A_n$, note that $t \mapsto t\,c$ is an odd, self-adjoint, regular unbounded multiplier on $C_0(\mathbb{R}, \mathbb{C}\ell_1)$. Therefore the functional calculus for this multiplier is a graded $*$-homomorphism $\widehat{S} \to C_0(\mathbb{R}, \mathbb{C}\ell_1)$. This in turn can be seen as a basepoint preserving map $S^1 \to \hom_{\mathrm{gr}}(\widehat{S}, \mathbb{C}\ell_1)$. Now consider

$$C_0(\mathbb{R}, \mathbb{C}\ell_1) \to C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K}) \quad ; \quad f \mapsto f \otimes (1 \otimes e) \ ,$$

where e is a rank 1-projection in $\mathbb{K}$. After concatenation, we obtain a graded $*$-homomorphism $\hat{\eta}_1\colon \widehat{S} \to C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})$ and from this a continuous map $\eta_1\colon S^1 \to \hom_{\mathrm{gr}}(\widehat{S}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})$. We define $\eta_0\colon S^0 \to KU^A_0$ by sending the non-basepoint to the counit of $\widehat{S}$. Now let

$$\hat{\eta}_n\colon \widehat{S} \to C_0(\mathbb{R}^n, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n}) \quad \text{with } \hat{\eta}_n = (\hat{\eta}_1 \widehat{\otimes} \dots \widehat{\otimes}\, \hat{\eta}_1) \circ \Delta_n \ ,$$

where $\Delta_n\colon \widehat{S} \to \widehat{S}^{\widehat{\otimes} n}$ is defined recursively by $\Delta_1 = \mathrm{id}_{\widehat{S}}$, $\Delta_2 = \Delta$ and $\Delta_n = (\Delta \mathbin{\widehat{\otimes}} \mathrm{id}) \circ \Delta_{n-1}$, for $n \geq 3$. This yields a well-defined map $\eta_n\colon S^n \to KU^A_n$.

Theorem 3.4.2. *Let A be a separable, unital, strongly self-absorbing C^*-algebra. The spaces $KU^A_\bullet$ together with the maps $\mu_{\bullet,\bullet}$ and $\eta_\bullet$ form a commutative symmetric ring spectrum with coefficients*

$$\pi_n(KU^A_\bullet) = K_n(A) \ .$$

Moreover, all structure maps $KU^A_n \to \Omega KU^A_{n+1}$ are weak homotopy equivalences for $n \geq 1$.

Proof. It is a consequence of the cocommutativity of Δ that Δ_n is Σ_n-invariant, i.e. for a permutation $\tau \in \Sigma_n$ and its induced action $\tau_* \colon \widehat{S}^{\widehat{\otimes} n} \to \widehat{S}^{\widehat{\otimes} n}$ we have $\tau_* \circ \Delta_n = \Delta_n$. If we let τ act on

$$C_0(\mathbb{R}^n, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n}) \cong (C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A \otimes \mathbb{K}))^{\widehat{\otimes} n}$$

by permuting the tensor factors, then we have that $\hat{\eta}_n$ is invariant under the action of Σ_n since

$$\tau_* \circ \hat{\eta}_n = \tau_* \circ (\hat{\eta}_1 \widehat{\otimes} \dots \widehat{\otimes}\, \hat{\eta}_1) \circ \Delta_n = (\hat{\eta}_1 \widehat{\otimes} \dots \widehat{\otimes}\, \hat{\eta}_1) \circ \tau_* \circ \Delta_n = \hat{\eta}_n \ ,$$

which proves that η_n is Σ_n-*equivariant*. The $\Sigma_m \times \Sigma_n$-equivariance of $\mu_{m,n}$ is clear from the definition of the Σ_{m+n}-action on KU^A_{m+n} and the symmetry of the graded tensor product. Associativity of $\mu_{\bullet,\bullet}$ (see Definition 3.2.1 (a)) is a direct consequence of the coassociativity of $\widehat{S}$ and the associativity of the graded tensor product. The map $\mu_{m,n}(\eta_m \wedge \eta_n)$ corresponds to the $*$-homomorphism

$$(\hat{\eta}_m \widehat{\otimes}\, \hat{\eta}_n) \circ \Delta = \underbrace{\hat{\eta}_1 \widehat{\otimes} \dots \widehat{\otimes}\, \hat{\eta}_1}_{n+m \text{ times}} \circ (\Delta_m \widehat{\otimes}\, \Delta_n) \circ \Delta \ .$$

By coassociativity we have $(\Delta_m \widehat{\otimes}\, \Delta_n) \circ \Delta = \Delta_{m+n}$. Therefore $(\hat{\eta}_m \widehat{\otimes}\, \hat{\eta}_n) \circ \Delta = \hat{\eta}_{m+n}$, which translates into the first compatibility condition of Definition 3.2.1. The other compatibility conditions are consequences of the fact that $\widehat{S}$ is counital. The commutativity of $\mu_{\bullet,\bullet}$ follows from the definition of the permutation action and the cocommutativity of the coalgebra structure on $\widehat{S}$. Thus, the spaces $KU^A_\bullet$ indeed form a commutative symmetric ring spectrum.

To see that the structure maps induce weak equivalences, observe that for $k \geq 1$

$$\pi_k(KU^A_n) = \pi_0(\Omega^k KU^A_n) = \pi_0(\hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes \mathbb{K} \otimes A)^{\widehat{\otimes} n})))$$

The algebra $\mathbb{C}\ell_1 \otimes \mathbb{K}$ with trivially graded $\mathbb{K}$ is isomorphic as a graded C^*-algebra to $\mathbb{C}\ell_1 \widehat{\otimes} \widehat{\mathbb{K}}$, where $\widehat{\mathbb{K}} = \widehat{\mathbb{K}}(H_+ \oplus H_-)$ is equipped with its standard even grading. Therefore

$$C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes \mathbb{K} \otimes A)^{\widehat{\otimes} n}) \cong C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n}) \widehat{\otimes}\, \widehat{\mathbb{K}}^{\widehat{\otimes} n} \cong C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n}) \widehat{\otimes}\, \widehat{\mathbb{K}}$$

and by [141, Theorem 4.7]

$$\pi_0(\hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n}) \widehat{\otimes}\, \widehat{\mathbb{K}}) \cong KK(\mathbb{C}, C_0(\mathbb{R}^k, (\mathbb{C}\ell_1 \otimes A)^{\widehat{\otimes} n})) \cong K_{k-n}(A)$$

where we used Bott periodicity and the isomorphism $A^{\otimes n} \cong A$ in the last map. The structure map $KU^A_n \to \Omega KU^A_{n+1}$ is now given by exterior multiplication with the class in $KK(\mathbb{C}, C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes A))$ represented by $\widehat{\eta}_1$. But by definition this is a combination of the Bott element together with the map $A \otimes \mathbb{K} \to (A \otimes \mathbb{K})^{\otimes 2}$ that sends a to $a \otimes (1 \otimes e)$, which is homotopic to an isomorphism, since A is *strongly* self-absorbing. Thus, both operations induce isomorphisms on K-theory and therefore also on all homotopy groups. Finally, we have for $n \in \mathbb{Z}$

$$\pi_n(KU^A_\bullet) = \operatorname{colim}_{k\to\infty} \pi_{n+k}(KU^A_k) \cong K_n(A) \ . \qquad \square$$

Remark 3.4.3. As can be seen from the above proof, the fact that A is *strongly* self-absorbing (and not just self-absorbing, which would be $A \cong A \otimes A$) is important for the spectrum to be positive fibrant.

Remark 3.4.4. Note that the spectrum $KU^A_\bullet$ is not connective. However, the $\mathcal{I}$-monoid $\Omega^\infty(R)^*$ of a symmetric ring spectrum R only "sees" the connective cover. In fact, $\Omega^\infty(R)_{h\mathcal{I}}$ is its underlying infinite loop space. This was already remarked in [123, Section 1.6] and was the motivation for the introduction of $\mathcal{J}$-spaces to also capture periodic phenomena.

3.4.2 The Eckmann-Hilton $\mathcal{I}$-group G_A

Let A be a separable, unital, strongly self-absorbing algebra and let $G_A(\mathbf{n}) = \operatorname{Aut}((A\otimes\mathbb{K})^{\otimes n})$ (with $(A\otimes\mathbb{K})^{\otimes 0} := \mathbb{C}$, such that $G_A(\mathbf{0})$ is the trivial group). Note that $\sigma \in \Sigma_n$ acts on $G_A(\mathbf{n})$ by mapping $g \in G_A(\mathbf{n})$ to $\sigma \circ g \circ \sigma^{-1}$, where σ permutes the tensor factors of $(A \otimes \mathbb{K})^{\otimes n}$. Let $g \in G_A(\mathbf{m})$ and $\alpha\colon \mathbf{m} \to \mathbf{n}$. We can enhance G_A to a functor $G_A\colon \mathcal{I} \to \mathcal{G}rp_*$ via

$$\alpha_*(g) = \bar{\alpha} \circ (\mathrm{id}_{(A\otimes\mathbb{K})^{\otimes(n-m)}} \otimes g) \circ \bar{\alpha}^{-1}$$

with $\bar{\alpha}$ as explained after Definition 3.2.2. Moreover, G_A is an $\mathcal{I}$-monoid via

$$\mu_{m,n}\colon G_A(\mathbf{m}) \times G_A(\mathbf{n}) \to G_A(\mathbf{m} \sqcup \mathbf{n}) \quad ; \quad (g, h) \mapsto g \otimes h \ .$$

Note that $\mu_{m,0}$ and $\mu_{0,n}$ are induced by the canonical isomorphisms $\mathbb{C}\otimes(A\otimes\mathbb{K})^{\otimes n} \cong (A\otimes\mathbb{K})^{\otimes n}$ and $(A \otimes \mathbb{K})^{\otimes m} \otimes \mathbb{C} \cong (A \otimes \mathbb{K})^{\otimes m}$ respectively.

Theorem 3.4.5. *G_A as defined above is a stable EH-$\mathcal{I}$-group with compatible inverses in the sense of Definition 3.3.1.*

Proof. The commutativity of the Eckmann-Hilton diagram in Definition 3.3.1 is a consequence of $(g \otimes h) \cdot (g' \otimes h') = (g \cdot g') \otimes (h \cdot h')$ for $g, g' \in G_A(\mathbf{m})$ and $h, h' \in G_A(\mathbf{n})$. Well-pointedness of $G_A(\mathbf{n})$ was proven in [47, Prop. 2.6]. To see that non-initial morphisms are mapped to homotopy equivalences, first observe that G_A maps permutations to homeomorphisms, therefore it suffices to check that

$$\operatorname{Aut}(A \otimes \mathbb{K}) \to \operatorname{Aut}((A \otimes \mathbb{K})^{\otimes n}) \quad ; \quad g \mapsto g \otimes \mathrm{id}_{(A\otimes\mathbb{K})^{\otimes(n-1)}}$$

is a homotopy equivalence. But the target space is homeomorphic to $\mathrm{Aut}(A \otimes \mathbb{K})$ via conjugation with a suitable isomorphism. Altogether it is enough to check that we can find an isomorphism $\psi \colon A \otimes \mathbb{K} \to (A \otimes \mathbb{K})^{\otimes 2}$ such that

$$\mathrm{Aut}(A \otimes \mathbb{K}) \to \mathrm{Aut}(A \otimes \mathbb{K}) \quad ; \quad g \mapsto \psi^{-1} \circ (g \otimes \mathrm{id}_{A\otimes\mathbb{K}}) \circ \psi$$

is homotopic to $\mathrm{id}_{\mathrm{Aut}(A\otimes\mathbb{K})}$. Let $\hat{\psi} \colon I \times (A \otimes \mathbb{K}) \to (A \otimes \mathbb{K})^{\otimes 2}$ be the homotopy of [47, Thm. 2.5] and let $\psi = \hat{\psi}(\frac{1}{2})$. Then

$$H \colon I \times \mathrm{Aut}(A \otimes \mathbb{K}) \to \mathrm{Aut}(A \otimes \mathbb{K}) \ ; \ g \mapsto \begin{cases} \hat{\psi}(\frac{t}{2})^{-1} \circ (g \otimes \mathrm{id}_{A\otimes\mathbb{K}}) \circ \hat{\psi}(\frac{t}{2}) & \text{for } 0 < t \leq 1 \\ g & \text{for } t = 0 \end{cases}$$

satisfies the conditions.

Let $g \in G_A(\mathbf{m})$. To see that there is a path connecting $(\iota_m \sqcup \mathrm{id}_{\mathbf{m}})_*(g) = \mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}} \otimes g$ to $g \otimes \mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}}$, observe that $\tau_A \colon A \otimes A \to A \otimes A$ with $\tau_A(a \otimes b) = b \otimes a$ is homotopic to the identity by the contractibility of $\mathrm{Aut}(A \otimes A) \cong \mathrm{Aut}(A)$ [47, Thm. 2.3]. Similarly, there is a homotopy between $\tau_{\mathbb{K}} \colon \mathbb{K} \otimes \mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$ with $\tau_{\mathbb{K}}(S \otimes T) = T \otimes S$ and the identity since $\mathrm{Aut}(\mathbb{K} \otimes \mathbb{K}) \cong PU(H)$ is path-connected.

Thus, we obtain a homotopy between the identity on $(A \otimes \mathbb{K})^{\otimes m} \otimes (A \otimes \mathbb{K})^{\otimes m}$ and the corresponding switch automorphism $\tau_m \colon (A\otimes\mathbb{K})^{\otimes m} \otimes (A\otimes\mathbb{K})^{\otimes m} \to (A\otimes\mathbb{K})^{\otimes m} \otimes (A\otimes\mathbb{K})^{\otimes m}$. But, $\tau_m \circ (\mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}} \otimes g) \circ \tau_m^{-1} = g \otimes \mathrm{id}_{(A\otimes\mathbb{K})^{\otimes m}}$. □

Recall that $KU_n^A = \hom_{\mathrm{gr}}(\widehat{S}, (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n})$. There is a unique isomorphism, which preserves the order of factors of the same type:

$$\theta_n \colon (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n} \to (\mathbb{C}\ell_1)^{\widehat{\otimes} n} \otimes (A \otimes \mathbb{K})^{\otimes n} \ ,$$

where – as above – the grading on both sides arises from the grading of $\mathbb{C}\ell_1$ and $A \otimes \mathbb{K}$ is trivially graded. In particular, if $\sigma \in \Sigma_n$ is a permutation and $\sigma_* \colon (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n} \to (\mathbb{C}\ell_1 \otimes A \otimes \mathbb{K})^{\widehat{\otimes} n}$ is the operation permuting the factors of the graded tensor product, then

$$\theta_n \circ \sigma_* = (\sigma_*^{\mathbb{C}\ell_1} \otimes \sigma_*^{A\otimes\mathbb{K}}) \circ \theta_n \tag{3.1}$$

where $\sigma_*^{\mathbb{C}\ell_1} \colon (\mathbb{C}\ell_1)^{\widehat{\otimes} n} \to (\mathbb{C}\ell_1)^{\widehat{\otimes} n}$ and $\sigma_*^{A\otimes\mathbb{K}} \colon (A \otimes \mathbb{K})^{\otimes n} \to (A \otimes \mathbb{K})^{\otimes n}$ are the corresponding permutations. Thus, we can define an action of $G_A(\mathbf{n})$ on KU_n^A via

$$\kappa_n \colon G_A(\mathbf{n}) \times KU_n^A \to KU_n^A \quad ; \quad (g, \varphi) \mapsto \theta_n^{-1} \circ (\mathrm{id}_{\mathbb{C}\ell_1^{\widehat{\otimes} n}} \otimes g) \circ \theta_n \circ \varphi$$

Theorem 3.4.6. *Let $A \neq \mathbb{C}$ be a separable, unital, strongly self-absorbing C^*-algebra. Then the EH-$\mathcal{I}$-group G_A associated to A acts on the commutative symmetric ring spectrum $KU_\bullet^A$*

via $\kappa_\bullet$ as defined above. We obtain a map of very special Γ-spaces $\Gamma(G_A) \to \Gamma(\Omega^\infty(KU^A)^*)$, which induces an isomorphism on all homotopy groups π_n of the associated spectra with $n > 0$ and the inclusion $K_0(A)_+^\times \to K_0(A)^\times$ on π_0. If A is purely infinite [119, Def. 4.1.2], this is an equivalence in the stable homotopy category.

Proof. It is clear that κ_n preserves the basepoint of KU^A_n. The Σ_n-equivariance is a direct consequence of (3.1). Indeed, we have for a permutation $\sigma \in \Sigma_n$, $g \in G_A(\mathbf{n})$ and $\varphi \in KU^A_n$:

$$\begin{aligned}
&\kappa_n(\sigma^{A\otimes\mathbb{K}} \circ g \circ (\sigma^{A\otimes\mathbb{K}})^{-1}, \sigma \circ \varphi) \\
&= \theta_n^{-1} \circ (\mathrm{id} \otimes (\sigma^{A\otimes\mathbb{K}} \circ g \circ (\sigma^{A\otimes\mathbb{K}})^{-1})) \circ \theta_n \circ \sigma \circ \varphi \\
&= \theta_n^{-1} \circ (\sigma^{\mathbb{C}\ell_1} \otimes \sigma^{A\otimes\mathbb{K}}) \circ (\mathrm{id} \otimes g) \circ (\sigma^{\mathbb{C}\ell_1} \otimes \sigma^{A\otimes\mathbb{K}})^{-1} \circ \theta_n \circ \sigma \circ \varphi \\
&= \sigma \circ \theta_n^{-1} \circ (\mathrm{id} \otimes g) \circ \theta_n \circ \varphi = \sigma \circ \kappa_n(g, \varphi) \ ,
\end{aligned}$$

where we omitted the stars from the notation.

Let $\tau\colon \mathbb{C}\ell_1^{\widehat{\otimes} m} \otimes (A \otimes \mathbb{K})^{\otimes m} \mathbin{\widehat{\otimes}} \mathbb{C}\ell_1^{\widehat{\otimes} n} \otimes (A \otimes \mathbb{K})^{\otimes n} \to \mathbb{C}\ell_1^{\widehat{\otimes} m+n} \mathbin{\widehat{\otimes}} (A \otimes \mathbb{K})^{\otimes m+n}$ be the permutation of the two middle factors, then $\tau \circ (\theta_m \mathbin{\widehat{\otimes}} \theta_n) = \theta_{m+n}$. This implies that

$$(\kappa_m(g, \varphi) \mathbin{\widehat{\otimes}} \kappa_n(h, \psi)) \circ \Delta = \kappa_{m+n}(g \otimes h,\, (\varphi \mathbin{\widehat{\otimes}} \psi) \circ \Delta)$$

for $g \in G_A(\mathbf{m}), h \in G_A(\mathbf{n})$, $\varphi \in KU^A_m, \psi \in KU^A_n$, which is the compatibility condition in Definition 3.3.7 (ii). The same argument shows that for $l + m = n$, $g \in G(\mathbf{m})$ and $\varphi \in KU^A_m$

$$(\widehat{\eta}_l \mathbin{\widehat{\otimes}} \kappa_m(g, \varphi)) \circ \Delta = \kappa_n(\mathrm{id} \otimes g, (\widehat{\eta}_l \mathbin{\widehat{\otimes}} \varphi) \circ \Delta) \ ,$$

which is the crucial observation to see that diagram (iii) in Definition 3.3.7 commutes.

Thus, we have proven that $G = G_A$ acts on the spectrum KU^A. By Theorem 3.3.8 together with Theorem 3.3.6 we obtain a map of Γ-spaces

$$\Gamma(B_\nu G_A) \to B_\mu \Gamma(G_A) \to B_\mu \Gamma(\Omega^\infty(KU^A)^*)$$

where the first map is a strict equivalence. We see from Lemma 3.3.3 that $G_{h\mathcal{I}}$ is in fact a grouplike topological monoid, i.e. $G_{h\mathcal{I}} \to \Omega B_\mu G_{h\mathcal{I}}$ is a homotopy equivalence. Thus, to finish the proof, we only need to check that $G_{h\mathcal{I}} \to GL_1(KU^A) = \Omega^\infty(KU^A)^*_{h\mathcal{I}}$ has the desired properties. Consider the diagram

$$\begin{tikzcd}
G_{h\mathcal{I}} \arrow[r] & GL_1(KU^A) \\
G(\mathbf{1}) \arrow[r] \arrow[u, "\simeq"] & \Omega^\infty(KU^A)^*(\mathbf{1}) \arrow[u, "\simeq"']
\end{tikzcd}$$

where the vertical maps are given by the inclusions into the zero skeleton of the homotopy colimit. The latter are equivalences by [124, Lemma 2.1]. By this lemma, we also see that $\pi_0(GL_1(KU^A)) = GL_1(\pi_0(KU^A)) = GL_1(K_0(A)) = K_0(A)^\times$. It remains to be seen that

$$\Theta\colon \mathrm{Aut}(A\otimes\mathbb{K}) \to \Omega KU_1^A = \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R},\mathbb{C}\ell_1)\otimes A\otimes\mathbb{K}) \quad ; \quad g\mapsto \kappa_1(g,\widehat{\eta}_1)$$

induces an isomorphism on all homotopy groups π_n with $n>0$ and the inclusion $K_0(A)_+^\times \to K_0(A)$ on π_0. The basepoint of the target space is now given by $\widehat{\eta}_1$ instead of the zero homomorphism. Θ fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}(A\otimes\mathbb{K}) \arrow[rr, "\Theta"] \arrow[dr, "\Phi"'] & & \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R},\mathbb{C}\ell_1)\otimes A\otimes\mathbb{K}) \\
& \mathcal{P}r(A\otimes\mathbb{K})^\times \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

with $\Phi(g) = g(1\otimes e)$ and $\Psi(p) = \widetilde{\eta}_1\otimes p$, where $\widetilde{\eta}_1 \in \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R},\mathbb{C}\ell_1))$ arises from the functional calculus of the operator described after Definition 3.4.1. It was shown in [47, Thm. 2.16, Thm. 2.5] that Φ is a homotopy equivalence. Let $\epsilon\in\hom_{\mathrm{gr}}(\widehat{S},\mathbb{C})$ be the counit of $\widehat{S}$. The map Ψ factors as

$$\Psi\colon \mathcal{P}r(A\otimes\mathbb{K})^\times \to \hom_{\mathrm{gr}}(\widehat{S}, A\mathbin{\widehat{\otimes}}\widehat{\mathbb{K}}) \to \hom_{\mathrm{gr}}(\widehat{S}, C_0(\mathbb{R},\mathbb{C}\ell_1)\otimes A\otimes\mathbb{K})$$

where the first map sends a projection p to $\epsilon\cdot\left(\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\right)$ and the second sends φ to $\varphi\mathbin{\widehat{\otimes}}\widetilde{\eta}_1\circ\Delta$ and applies the graded isomorphism to shift the grading to $C_0(\mathbb{R},\mathbb{C}\ell_1)$ only. Since the second map induces multiplication with the Bott element, it is an isomorphism on π_0. It was proven in [47, Cor. 2.17] that $\pi_0(\mathcal{P}r(A\otimes\mathbb{K})^\times)\cong K_0(A)_+^\times$. The discussion after the proof of [141, Thm. 4.7] shows that the first map induces the inclusion $K_0(A)_+^\times\to K_0(A)$ on π_0.

Let B be a graded, σ-unital C^*-algebra and define $K_n'(B)$ to be the kernel of the map $K'(C(S^n)\otimes B)\to K'(B)$ induced by evaluation at the basepoint. Here, we used the notation $K'(B) = \pi_0(\hom_{\mathrm{gr}}(\widehat{S}, B\mathbin{\widehat{\otimes}}\widehat{K}))$ introduced in [141]. The five lemma shows that $K_n'(B)$ is in fact isomorphic to $K_n(B)$, if we identify the latter with the kernel $K_0(C(S^n)\otimes B)\to K_0(B)$. For $n>0$ we have the commutative diagram

$$\begin{tikzcd}
\pi_n(\mathcal{P}r(A\otimes\mathbb{K})^\times, 1\otimes e) \arrow[r, "\Psi_*"] \arrow[d, "\cong"'] & \pi_n(\Omega KU_1^A, \widehat{\eta}_1) \arrow[d, "\cong"] \\
K_n(A) \arrow[r, "\cong"'] & K_n'(C_0(\mathbb{R},\mathbb{C}\ell_1)\otimes A)
\end{tikzcd}$$

where the lower horizontal map sends $[p]-[q]\in K_n(A) = \ker(K_0(C(S^n)\otimes A)\to K_0(A))$

to $\left[\widetilde{\eta}_1 \mathbin{\widehat{\otimes}} \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right] \in K_n'(C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A)$. The same argument as for π_0 above shows that this is an isomorphism. Every element $\gamma\colon (S^n, x_0) \to (\mathcal{P}r(A \otimes \mathbb{K}), 1 \otimes e)$ induces a projection $p_\gamma \in C(S^n) \otimes A \otimes \mathbb{K}$. The vertical map on the left sends $[\gamma]$ to $[p_\gamma] - [1_{C(S^n)} \otimes 1 \otimes e] \in K_0(C(S^n, x_0) \otimes A) \cong K_n(A)$. This map is an isomorphism by Bott periodicity. Finally, we can consider $\gamma'\colon S^n \to \Omega KU_1^A$ as an element $\varphi_{\gamma'} \in \hom_{\mathrm{gr}}(\widehat{S}, C(S^n) \otimes C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A \otimes \mathbb{K})$ ($\mathbb{K}$ trivially graded!). The vertical map on the right hand side sends $[\gamma']$ to $\left[\begin{pmatrix} \varphi_{\gamma'} & 0 \\ 0 & 1_{C(S^n)} \otimes \widehat{\eta}_1 \end{pmatrix}\right] \in K_n'(C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes A)$. It corresponds to the 'subtraction of 1', i.e. it corrects the basepoint by shifting back to the component of the zero homomorphism. Its inverse is given by $\psi \mapsto \psi \oplus \left(\begin{smallmatrix} 1_{C(S^n)} \otimes \widehat{\eta}_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $\oplus$ is the addition operation described in [141].

If A is purely infinite, we have $K_0(A)_+^\times = K_0(A)^\times$, which implies the last statement. This finishes the proof. $\square$

3.4.3 Applications

In this section we apply the above results to some examples of strongly self-absorbing C^*-algebras starting with $\mathcal{O}_\infty$ and $\mathcal{Z}$. It is a consequence of [139, 150] that any unital $*$-homomorphism between two strongly self-absorbing C^*-algebras is unique up to asymptotic unitary equivalence. From a categorical point of view, the Jiang-Su algebra $\mathcal{Z}$ is characterized by the property that it is the unique infinite dimensional strongly self-absorbing C^*-algebra which maps into any other infinite dimensional strongly self-absorbing C^*-algebra by a unital $*$-homomorphism. In this sense it is initial among those [150]. Likewise the infinite Cuntz algebra $\mathcal{O}_\infty$ is the unique purely infinite strongly self-absorbing C^*-algebra that maps unitally into any other purely infinite strongly self-absorbing C^*-algebra. Alternatively, $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by countably infinitely many generators s_i that satisfy the relations $s_i^* s_j = \delta_{ij}\, 1$. For any locally compact Hausdorff space X the unit homomorphisms $\mathbb{C} \to \mathcal{O}_\infty$ and $\mathbb{C} \to \mathcal{Z}$ induce natural isomorphisms of multiplicative cohomology theories $K^0(X) = K_0(C(X)) \to K_0(C(X) \otimes \mathcal{Z})$ and $K^0(X) \to K_0(C(X) \otimes \mathcal{O}_\infty)$.

Theorem 3.4.7. *The very special Γ-spaces $\Gamma(G_{\mathcal{O}_\infty})$ and $\Gamma(\Omega^\infty(KU)^*)$ are strictly equivalent, which implies that the spectrum associated to $\Gamma(G_{\mathcal{O}_\infty})$ is equivalent to $gl_1(KU)$ in the stable homotopy category. In particular, $B\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})$ is weakly homotopy equivalent to $BBU_\otimes \times B(\mathbb{Z}/2\mathbb{Z})$. Likewise, the spectrum associated to $\Gamma(G_{\mathcal{Z}})$ is equivalent to $sl_1(KU)$, the 0-connected cover of $gl_1(KU)$, and $B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})$ is weakly homotopy equivalent to $BBU_\otimes$.*

Proof. The unit homomorphism $\mathbb{C} \to \mathcal{O}_\infty$ yields $\Gamma(\Omega^\infty(KU^{\mathbb{C}})^*) \to \Gamma(\Omega^\infty(KU^{\mathcal{O}_\infty})^*)$. To see that this is a strict equivalence, it suffices to check that $GL_1(KU) \to GL_1(KU^{\mathcal{O}_\infty})$ is a weak equivalence. Let X be a finite CW-complex. The isomorphism

$$[X, \Omega^1(KU_1^{\mathcal{O}_\infty})] \cong K_0(C(X) \otimes \mathcal{O}_\infty)$$

constructed above restricts to $[X, GL_1(KU^{\mathcal{O}_\infty})] \cong GL_1(K_0(C(X) \otimes \mathcal{O}_\infty))$ and similarly with $\mathbb{C}$ instead of $\mathcal{O}_\infty$. The composition

$$GL_1(K^0(X)) \cong [X, GL_1(KU)] \to [X, GL_1(KU^{\mathcal{O}_\infty})] \cong GL_1(K_0(C(X) \otimes \mathcal{O}_\infty))$$

is the restriction of the ring isomorphism $K^0(X) \to K_0(C(X) \otimes \mathcal{O}_\infty)$ to the invertible elements.

By Theorem 3.4.6 we obtain a strict equivalence $\Gamma(G_{\mathcal{O}_\infty}) \to \Gamma(\Omega^\infty(KU^{\mathcal{O}_\infty})^*)$. Therefore the zig-zag $\Gamma(G_{\mathcal{O}_\infty}) \to \Gamma(\Omega^\infty(KU^{\mathcal{O}_\infty})^*) \leftarrow \Gamma(\Omega^\infty(KU)^*)$ proves the first claim. From this, we get a weak equivalence between $BGL_1(KU) \simeq BBU_\otimes \times B(\mathbb{Z}/2\mathbb{Z})$ and $B_\nu G_A(\mathbf{1}) = B\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})$ using Lemma 3.3.5 and the stability of G_A.

The unit homomorphism $\mathbb{C} \to \mathcal{Z}$ yields $\Gamma(\Omega^\infty(KU^{\mathbb{C}})^*) \to \Gamma(\Omega^\infty(KU^{\mathcal{Z}})^*)$, which is again a strict equivalence by the same reasoning as above. By Theorem 3.4.6 the map $\Gamma(G_{\mathcal{Z}}) \to \Gamma(\Omega^\infty(KU^{\mathcal{Z}})^*)$ induces an isomorphism on π_k for $k > 0$ and corresponds to the inclusion $K_0(\mathcal{Z})_+^\times \to K_0(\mathcal{Z})^\times$ on π_0. But since $K_0(\mathcal{Z})_+^\times \cong (\mathbb{Z})_+^\times$ is trivial, the spectrum associated to $\Gamma(G_{\mathcal{Z}})$ corresponds to the 0-connected cover of the one associated to $\Gamma(\Omega^\infty(KU^{\mathcal{Z}})^*)$, which is $sl_1(KU)$. This implies $B\mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K}) \simeq BSL_1(KU) \simeq BBU_\otimes$ by Lemma 3.3.5. □

The UHF-algebra M_{p^∞} is constructed as an infinite tensor product of matrix algebras $M_p(\mathbb{C})$. It is separable, unital, strongly self-absorbing with $K_0(M_{p^\infty}) \cong \mathbb{Z}[\frac{1}{p}]$, $K_1(M_{p^\infty}) = 0$. Likewise, if we fix a prime p and choose a sequence $(d_j)_{j\in\mathbb{N}}$ such that each prime number except p appears in the sequence infinitely many times, we can recursively define $A_{n+1} = A_n \otimes M_{d_n}$, $A_0 = \mathbb{C}$. The direct limit $M_{(p)} = \lim A_n$ is a separable, unital, strongly self-absorbing C^*-algebra with $K_0(M_{(p)}) \cong \mathbb{Z}_{(p)}$ – the localization of the integers at p – and $K_1(M_{(p)}) = 0$.

Theorem 3.4.8. *The very special Γ-spaces $\Gamma(G_{M_{(p)}\otimes\mathcal{O}_\infty})$ and $\Gamma(\Omega^\infty(KU_{(p)})^*)$ are strictly equivalent, which implies that the spectrum associated to $\Gamma(G_{M_{(p)}\otimes\mathcal{O}_\infty})$ is equivalent to the spectrum $gl_1(KU_{(p)})$ in the stable homotopy category. In particular, $B\mathrm{Aut}(M_{(p)} \otimes \mathcal{O}_\infty \otimes \mathbb{K})$ is weakly homotopy equivalent to $BGL_1(KU_{(p)})$. The analogous statement for the localization* away *from p is also true if $M_{(p)}$ is replaced by M_{p^∞}.*

Proof. Consider the commutative symmetric ring spectrum $KU_\bullet^{M_{(p)}\otimes\mathcal{O}_\infty}$ with homotopy groups $\pi_{2k}(KU^{M_{(p)}\otimes\mathcal{O}_\infty}) = \mathbb{Z}_{(p)}$, $\pi_{2k+1}(KU^{M_{(p)}\otimes\mathcal{O}_\infty}) = 0$. Note that the unit homomorphism $\mathbb{C} \to M_{(p)} \otimes \mathcal{O}_\infty$ induces a map of spectra $KU^{\mathbb{C}} \to KU^{M_{(p)}\otimes\mathcal{O}_\infty}$, which is the localization map $\mathbb{Z} \to \mathbb{Z}_{(p)}$ on the non-zero coefficient groups. Let $S_{(p)}$ be the Moore spectrum, i.e. the commutative symmetric ring spectrum with $\widetilde{H}^1(S_{(p)}; \mathbb{Z}) \cong \mathbb{Z}_{(p)}$ and $\widetilde{H}^k(S_{(p)}; \mathbb{Z}) = 0$ for $k \neq 1$. We have $KU_{(p)} = KU \wedge S_{(p)}$ and $\pi_k(KU) \to \pi_k(KU_{(p)}) = \pi_k(KU) \otimes \mathbb{Z}_{(p)}$ is the localization map. It follows that $KU_{(p)} \to KU^{M_{(p)}\otimes\mathcal{O}_\infty} \wedge S_{(p)} \leftarrow KU^{M_{(p)}\otimes\mathcal{O}_\infty}$ is a zig-zag of

π_*-equivalences. Just as in Theorem 3.4.7 we show that it induces a zig-zag of strict equivalences $\Gamma(\Omega^\infty(KU_{(p)})^*) \to \Gamma(\Omega^\infty(KU^{M_{(p)}\otimes\mathcal{O}_\infty} \wedge S_{(p)})^*) \leftarrow \Gamma(\Omega^\infty(KU^{M_{(p)}\otimes\mathcal{O}_\infty})^*)$, which shows that $gl_1(KU_{(p)})$ is stably equivalent to $gl_1(KU^{M_{(p)}\otimes\mathcal{O}_\infty})$. But by Theorem 3.4.6 we have a strict equivalence $\Gamma(G_{M_{(p)}\otimes\mathcal{O}_\infty}) \to \Gamma(\Omega^\infty(KU^{M_{(p)}\otimes\mathcal{O}_\infty})^*)$. The proof for the localization away from p is completely analogous; therefore we omit it. □

In [47] the authors used a permutative category $\mathcal{B}_\otimes$ to show that $B\mathrm{Aut}(A\otimes\mathbb{K})$ carries an infinite loop space structure: The objects of $\mathcal{B}_\otimes$ are the natural numbers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\hom(m, n) = \{\alpha \in \hom((A\otimes\mathbb{K})^{\otimes m}, (A\otimes\mathbb{K})^{\otimes n}) \mid \alpha((1\otimes e)^{\otimes m}) \in GL_1(K_0((A\otimes\mathbb{K})^{\otimes n}))\}$, where $e \in \mathbb{K}$ is a rank 1-projection and $(1 \otimes e)^{\otimes 0} = 1 \in \mathbb{C}$. Since $\hom(0, 1)$ is non-empty, there is a stabilization θ of the object $1 \in \mathrm{obj}(\mathcal{B}_\otimes)$ by Lemma 3.3.10. In fact, we may choose $\theta(\iota_{\mathbf{1}}) \in \hom(\mathbb{C}, A \otimes \mathbb{K})$ to be $\theta(\lambda) = \lambda(1 \otimes e)$.

Theorem 3.4.9. *Let A be a separable, strongly self-absorbing C^*-algebra. Then there is a strict equivalence of Γ-spaces $\Gamma(B_\nu G_A) \to \Gamma(\mathcal{B}_\otimes)$. In particular, the induced infinite loop space structures on $B\mathrm{Aut}(A \otimes \mathbb{K})$ agree.*

Proof. Note that $\mathrm{Aut}_{\mathcal{B}_\otimes}(1) = \mathrm{Aut}(A \otimes \mathbb{K})$ and that the maps $B_\nu G_A(\mathbf{m}) = B\mathrm{Aut}((A \otimes \mathbb{K})^{\otimes m}) \to |N_\bullet\mathcal{B}_\otimes|$ are homotopy equivalences for $m > 0$. Thus, the statement follows from Theorem 3.3.13. □

The above identification may be used to prove theorems about bundles (and continuous fields) of strongly self-absorbing C^*-algebras using what is known about the unit spectrum of K-theory. As an example let X be a compact metrizable space and consider the cohomology group $[X, B\mathrm{Aut}_0(\mathcal{O}_\infty\otimes\mathbb{K})]$, where we use the notation of [47]. Note that the third Postnikov section of $B\mathrm{Aut}_0(\mathcal{O}_\infty\otimes\mathbb{K})$ is a $K(\mathbb{Z}, 3)$, let $B\mathrm{Aut}_0(\mathcal{O}_\infty\otimes\mathbb{K}) \to K(\mathbb{Z}, 3)$ be the corresponding map and denote by F its homotopy fiber. The composition $B\mathrm{Aut}(\mathbb{K}) \to B\mathrm{Aut}_0(\mathcal{O}_\infty\otimes\mathbb{K}) \to K(\mathbb{Z}, 3)$, where the first map is induced by the unit homomorphism $\mathbb{C} \to \mathcal{O}_\infty$, is a homotopy equivalence. Therefore we obtain a homotopy splitting

$$B\mathrm{Aut}(\mathbb{K}) \times F \xrightarrow{\simeq} B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$$

and a corresponding fibration $B\mathrm{Aut}(\mathbb{K}) \to B\mathrm{Aut}_0(\mathcal{O}_\infty\otimes\mathbb{K}) \to F$. The weak equivalence between $B\mathrm{Aut}_0(\mathcal{O}_\infty\otimes\mathbb{K})$ and $BBU_\otimes \simeq BBU(1)\times BBSU_\otimes$ identifies F with the corresponding homotopy fiber of the third Postnikov section of $BBU_\otimes$, which is $BBSU_\otimes$. Thus, we obtain a short exact sequence

$$0 \to [X, B\mathrm{Aut}(\mathbb{K})] \to [X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})] \to [X, F] \cong bsu^1_\otimes(X) \to 0$$

The following was proven by Gómez in [65, Theorem 5].

Theorem 3.4.10. *Let G be a compact Lie group. Then $bsu^1_{\otimes}(BG) = 0$.*

In light of our previous results, we obtain the Corollary:

Corollary 3.4.11. *Let X be a compact metrizable space, let G be a compact Lie group and let $P \to X$ be a principal G-bundle. Let $\alpha\colon G \to \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ be a continuous homomorphism. Then the associated $\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$-bundle $Q \to X$ given by $Q = P \times_\alpha \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ is isomorphic to $\widetilde{Q} \times_{\mathrm{Ad}} \mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})$ for a principal $PU(H)$-bundle $\widetilde{Q}$.*

Proof. The class $[Q] \in [X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]$ is the pullback of $[B\alpha] \in [BG, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]$ via the classifying map $f_P\colon X \to BG$ of P. The diagram

$$\begin{tikzcd}
 & & {[BG, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]} \arrow[r] \arrow[d] & {[BG, F] \cong bsu^1_{\otimes}(BG)} \arrow[d] & \\
0 \arrow[r] & {[X, B\mathrm{Aut}(\mathbb{K})]} \arrow[r] & {[X, B\mathrm{Aut}_0(\mathcal{O}_\infty \otimes \mathbb{K})]} \arrow[r] & {[X, F]} \arrow[r] & 0
\end{tikzcd}$$

shows that $[Q]$ is mapped to 0 in $[X, F] \cong bsu^1_{\otimes}(X)$, which implies the statement. □

4 The Brauer group in generalized Dixmier-Douady Theory

We have previously shown that the isomorphism classes of orientable locally trivial fields of C^*-algebras over a compact metrizable space X with fiber $D \otimes \mathbb{K}$, where D is a strongly self-absorbing C^*-algebra, form an abelian group under the operation of tensor product. Moreover this group is isomorphic to the first group $\bar{E}^1_D(X)$ of the (reduced) generalized cohomology theory associated to the unit spectrum of topological K-theory with coefficients in D. Here we show that all the torsion elements of the group $\bar{E}^1_D(X)$ arise from locally trivial fields with fiber $D \otimes M_n(\mathbb{C})$, $n \geq 1$, for all known examples of strongly self-absorbing C^*-algebras D. Moreover the Brauer group generated by locally trivial fields with fiber $D \otimes M_n(\mathbb{C})$, $n \geq 1$ is isomorphic to $Tor(\bar{E}^1_D(X))$.

4.1 Introduction

Let X be a compact metrizable space. Let $\mathbb{K}$ denote the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. It is well-known that $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$ and $M_n(\mathbb{C}) \otimes \mathbb{K} \cong \mathbb{K}$. Dixmier and Douady [55] showed that the isomorphism classes of locally trivial fields of C^*-algebras over X with fiber $\mathbb{K}$ form an abelian group under the operation of tensor product over $C(X)$ and this group is isomorphic to $H^3(X, \mathbb{Z})$. The torsion subgroup of $H^3(X, \mathbb{Z})$ admits the following description. Each element of $Tor(H^3(X, \mathbb{Z}))$ arises as the Dixmier-Douady class of a field A which is isomorphic to the stabilization $B \otimes \mathbb{K}$ of some locally trivial field of C^*-algebras B over X with all fibers isomorphic to $M_n(\mathbb{C})$ for some integer $n \geq 1$, see [67], [8].

In this paper we generalize this result to locally trivial fields with fiber $D \otimes \mathbb{K}$ where D is a strongly self-absorbing C^*-algebra [139]. For a C^*-algebra B, we denote by $\mathscr{C}_B(X)$ the isomorphism classes of locally trivial continuous fields of C^*-algebras over X with fibers isomorphic to B. The isomorphism classes of orientable locally trivial continuous fields is denoted by $\mathscr{C}^0_B(X)$, see Definition 4.2.1. We have shown in [47] that $\mathscr{C}_{D\otimes\mathbb{K}}(X)$ is an abelian group under the operation of tensor product over $C(X)$, and moreover, this group is isomorphic to the first group $E^1_D(X)$ of a generalized cohomology theory $E^*_D(X)$ which we have proven to be isomorphic to the theory associated to the unit spectrum of topological K-theory with coefficients in D, see [49]. Similarly $(\mathscr{C}^0_{D\otimes\mathbb{K}}(X), \otimes) \cong \bar{E}^1_D(X)$ where $\bar{E}^*_D(X)$ is the reduced theory associated to $E^*_D(X)$. For $D = \mathbb{C}$, we have, of course, $E^1_{\mathbb{C}}(X) \cong H^3(X, \mathbb{Z})$.

We consider the stabilization map $\sigma : \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \to (\mathscr{C}_{D\otimes\mathbb{K}}(X), \otimes) \cong E^1_D(X)$ given by $[A] \mapsto [A \otimes \mathbb{K}]$ and show that its image consists entirely of torsion elements. Moreover, if D is any of the known strongly self-absorbing C^*-algebras, we show that the stabilization map

$$\sigma : \bigcup_{n\geq 1} \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \to Tor(\bar{E}^1_D(X))$$

is surjective, see Theorem 4.2.10. In this situation we have $\mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \cong \mathscr{C}^0_{D\otimes M_n(\mathbb{C})}(X)$ by Lemma 4.2.1 and hence the image of the stabilization map is contained in the reduced group $\bar{E}^1_D(X)$. In analogy with the classic Brauer group generated by continuous fields of complex matrices $M_n(\mathbb{C})$ [67], we introduce a Brauer group $Br_D(X)$ for locally trivial fields of C*-algebras with fibers $M_n(D)$ for D a strongly self-absorbing C^*-algebra and establish an isomorphism $Br_D(X) \cong Tor(\bar{E}^1_D(X))$, see Theorem 4.2.15.

Our proof is new even in the classic case $D = \mathbb{C}$ whose original proof relies on an argument of Serre, see [67, Thm.1.6], [8, Prop.2.1]. In the cases $D = \mathcal{Z}$ or $D = \mathcal{O}_\infty$ the group $\bar{E}^1_D(X)$ is isomorphic to $H^1(X, BSU_\otimes)$, which appeared in [134], where its equivariant counterpart played a central role.

We introduced in [47] characteristic classes

$$\delta_0 : E^1_D(X) \to H^1(X, K_0(D)^\times_+) \quad \text{and} \quad \delta_k : E^1_D(X) \to H^{2k+1}(X, \mathbb{Q}), \quad k \geq 1.$$

If X is connected, then $\bar{E}^1_D(X) = \ker(\delta_0)$. We show that an element a belongs $Tor(E^1_D(X))$ if and only if $\delta_0(a)$ is a torsion element and $\delta_k(a) = 0$ for all $k \geq 1$.

In the last part of the paper we show that if A^{op} is the opposite C*-algebra of a locally trivial continuous field A with fiber $D \otimes \mathbb{K}$, then $\delta_k(A^{op}) = (-1)^k \delta_k(A)$ for all $k \geq 0$. This shows that in general $A \otimes A^{op}$ is not isomorphic to a trivial field, unlike what happens in the case $D = \mathbb{C}$. Similar arguments show that in general $[A^{op}]_{Br} \neq -[A]_{Br}$ in $Br_D(X)$ for $A \in \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X)$, see Example 4.3.5.

We would like to thank Ilan Hirshberg for prompting us to seek a refinement of Theorem 4.2.10 in the form of Theorem 4.2.11.

4.2 Background and main result

The class of strongly self-absorbing C^*-algebras was introduced by Toms and Winter [139]. They are separable unital C^*-algebras D singled out by the property that there exists an isomorphism $D \to D \otimes D$ which is unitarily homotopic to the map $d \mapsto d \otimes 1_D$ [51], [150].

If $n \geq 2$ is a natural number we denote by M_{n^∞} the UHF-algebra $M_n(\mathbb{C})^{\otimes\infty}$. If P is a nonempty set of primes, we denote by M_{P^∞} the UHF-algebra of infinite type $\bigotimes_{p\in P} M_{p^\infty}$. If P is the set of all primes, then M_{P^∞} is the universal UHF-algebra, which we denote by $M_{\mathbb{Q}}$.

The class $\mathcal{D}_{pi}$ of all purely infinite strongly self-absorbing C^*-algebras that satisfy the Universal Coefficient Theorem in KK-theory (UCT) was completely described in [139]. $\mathcal{D}_{pi}$ consists of the Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$ and of all C^*-algebras $M_{P^\infty} \otimes \mathcal{O}_\infty$ with P an arbitrary set of primes. Let $\mathcal{D}_{qd}$ denote the class of strongly self-absorbing C^*-algebras which satisfy the UCT and which are quasidiagonal. A complete description of $\mathcal{D}_{qd}$ has become possible due to the recent results of Matui and Sato [98, Cor. 6.2] that build on results of Winter [151], and Lin and Niu [91]. Thus $\mathcal{D}_{qd}$ consists of $\mathbb{C}$, the Jiang-Su algebra $\mathcal{Z}$ and all UHF-algebras M_{P^∞} with P an arbitrary set of primes. The class $\mathcal{D} = \mathcal{D}_{qd} \cup \mathcal{D}_{pi}$ contains all known examples of strongly self-absorbing C^*-algebras. It is closed under tensor products. If D is strongly self-absorbing, then $K_0(D)$ is a unital commutative ring. The group of positive invertible elements of $K_0(D)$ is denoted by $K_0(D)_+^\times$.

Let B be a C^*-algebra. We denote by $\mathrm{Aut}_0(B)$ the path component of the identity of $\mathrm{Aut}(B)$ endowed with the point-norm topology. Recall that we denote by $\mathscr{C}_B(X)$ the isomorphism classes of locally trivial continuous fields over X with fibers isomorphic to B. The structure group of $A \in \mathscr{C}_B(X)$ is $\mathrm{Aut}(B)$, and A is in fact given by a principal $\mathrm{Aut}(B)$-bundle which is determined up to an isomorphism by an element of the homotopy classes of continuous maps from X to the classifying space of the topological group $\mathrm{Aut}(B)$, denoted by $[X, B\mathrm{Aut}(B)]$.

Definition 4.2.1. A locally trivial continuous field A of C^*-algebras with fiber B is *orientable* if its structure group can be reduced to $\mathrm{Aut}_0(B)$, in other words if A is given by an element of $[X, B\mathrm{Aut}_0(B)]$.

The corresponding isomorphism classes of orientable and locally trivial fields is denoted by $\mathscr{C}_B^0(X)$.

Lemma 4.2.2. *Let D be a strongly self-absorbing C^*-algebra satisfying the UCT. Then* $\mathrm{Aut}(M_n(D)) = \mathrm{Aut}_0(M_n(D))$ *for all $n \geq 1$ and hence* $\mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \cong \mathscr{C}^0_{D\otimes M_n(\mathbb{C})}(X)$.

Proof. First we show that for any $\beta \in \mathrm{Aut}(D \otimes M_n(\mathbb{C}))$ there exist $\alpha \in \mathrm{Aut}(D)$ and a unitary $u \in D \otimes M_n(\mathbb{C})$ such that $\beta = u(\alpha \otimes \mathrm{id}_{M_n(\mathbb{C})})u^*$. Let $e_{11} \in M_n(\mathbb{C})$ be the rank-one projection that appears in the canonical matrix units (e_{ij}) of $M_n(\mathbb{C})$ and let 1_n be the unit of $M_n(\mathbb{C})$. Then $n[1_D \otimes e_{11}] = [1_D \otimes 1_n]$ in $K_0(D)$ and hence $n[\beta(1_D \otimes e_{11})] = n[1_D \otimes e_{11}]$ in $K_0(D)$. Under the assumptions of the lemma, it is known that $K_0(D)$ is torsion free (by [139]) and that D has cancellation of full projections by [150] and [120]. It follows that there is a partial isometry $v \in D \otimes M_n(\mathbb{C})$ such that $v^*v = 1_D \otimes e_{11}$ and $vv^* = \beta(1_D \otimes e_{11})$. Then $u = \sum_{i=1}^n \beta(1_D \otimes e_{i1})v(1_D \otimes e_{1i}) \in D \otimes M_n(\mathbb{C})$ is a unitary such that the automorphism $u^*\beta\, u$

acts identically on $1_D \otimes M_n(\mathbb{C})$. It follows that $u^*\beta\, u = \alpha \otimes \mathrm{id}_{M_n(\mathbb{C})}$ for some $\alpha \in \mathrm{Aut}(D)$. Since both $U(D \otimes M_n(\mathbb{C}))$ and $\mathrm{Aut}(D)$ are path connected by [139], [120] and respectively [51] we conclude that $\mathrm{Aut}(D \otimes M_n(\mathbb{C}))$ is path-connected as well. □

Let us recall the following results contained in Cor. 3.7, Thm. 3.8 and Cor. 3.9 from [47]. Let D be a strongly self-absorbing C^*-algebra.

(1) The classifying spaces $B\mathrm{Aut}(D \otimes \mathbb{K})$ and $B\mathrm{Aut}_0(D \otimes \mathbb{K})$ are infinite loop spaces giving rise to generalized cohomology theories $E^*_D(X)$ and respectively $\bar{E}^*_D(X)$.

(2) The monoid $(\mathscr{C}_{D\otimes\mathbb{K}}(X), \otimes)$ is an abelian group isomorphic to $E^1_D(X)$. Similarly, the monoid $(\mathscr{C}^0_{D\otimes\mathbb{K}}(X), \otimes)$ is a group isomorphic to $\bar{E}^1_D(X)$. In both cases the tensor product is understood to be over $C(X)$.

(3) $E^1_{M_\mathbb{Q}}(X) \cong H^1(X, \mathbb{Q}^\times_+) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})$,
$E^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})$,

(4) $\bar{E}^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong \bar{E}^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})$.

(5) If D satisfies the UCT then $D \otimes M_\mathbb{Q} \otimes \mathcal{O}_\infty \cong M_\mathbb{Q} \otimes \mathcal{O}_\infty$, by [139]. Therefore the tensor product operation $A \mapsto A \otimes M_\mathbb{Q} \otimes \mathcal{O}_\infty$ induces maps

$$\mathscr{C}_{D\otimes\mathbb{K}}(X) \to \mathscr{C}_{M_\mathbb{Q}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X), \quad \mathscr{C}^0_{D\otimes\mathbb{K}}(X) \to \mathscr{C}^0_{M_\mathbb{Q}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \quad \text{and hence maps}$$

$$E^1_D(X) \xrightarrow{\delta} E^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q}),$$
$$\delta(A) = (\delta^s_0(A), \delta_1(A), \delta_2(A), \dots), \quad \delta_k(A) \in H^{2k+1}(X, \mathbb{Q}),$$

$$\bar{E}^1_D(X) \xrightarrow{\bar{\delta}} \bar{E}^1_{M_\mathbb{Q}\otimes\mathcal{O}_\infty}(X) \cong \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q}),$$
$$\bar{\delta}(A) = (\delta_1(A), \delta_2(A), \dots), \quad \delta_k(A) \in H^{2k+1}(X, \mathbb{Q}).$$

The invariants $\delta_k(A)$ are called the rational characteristic classes of the continuous field A, see [47, Def.4.6]. The first class δ^s_0 lifts to a map $\delta_0 : E^1_D(X) \to H^1(X, K_0(D)^\times_+)$ induced by the morphism of groups $\mathrm{Aut}(D \otimes \mathbb{K}) \to \pi_0(\mathrm{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^\times_+$. $\delta_0(A)$ represents the obstruction to reducing the structure group of A to $\mathrm{Aut}_0(D \otimes \mathbb{K})$.

Proposition 4.2.3. *A continuous field $A \in \mathscr{C}_{D\otimes\mathbb{K}}(X)$ is orientable if and only if $\delta_0(A) = 0$. If X is connected, then $\bar{E}^1_D(X) \cong \ker(\delta_0)$.*

Proof. Let us recall from [47, Cor. 2.19] that there is an exact sequence of topological groups

$$1 \to \mathrm{Aut}_0(D \otimes \mathbb{K}) \to \mathrm{Aut}(D \otimes \mathbb{K}) \xrightarrow{\pi} K_0(D)_+^\times \to 1. \tag{4.1}$$

The map π takes an automorphism α to $[\alpha(1_D \otimes e)]$ where $e \in \mathbb{K}$ is a rank-one projection. If G is a topological group and H is a normal subgroup of G such that $H \to G \to G/H$ is a principal H-bundle, then there is a homotopy fibre sequence $G/H \to BH \to BG \to B(G/H)$ and hence an exact sequence of pointed sets $[X, G/H] \to [X, BH] \to [X, BG] \to [X, B(G/H)]$. In particular, in the case of the fibration (4.1) we obtain

$$[X, K_0(D)_+^\times] \to [X, B\mathrm{Aut}_0(D \otimes \mathbb{K})] \to [X, B\mathrm{Aut}(D \otimes \mathbb{K})] \xrightarrow{\delta_0} H^1(X, K_0(D)_+^\times). \tag{4.2}$$

A continuous field $A \in \mathscr{C}^0_{D\otimes\mathbb{K}}(X)$ is associated to a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle whose classifying map gives a unique element in $[X, B\mathrm{Aut}(D\otimes\mathbb{K})]$ whose image in $H^1(X, K_0(D)_+^\times)$ is denoted by $\delta_0(A)$. It is clear from (4.2) that the class $\delta_0(A) \in H^1(X, K_0(D)_+^\times)$ represents the obstruction for reducing this bundle to a principal $\mathrm{Aut}_0(D\otimes\mathbb{K})$-bundle. If X is connected, $[X, K_0(D)_+^\times] = \{*\}$ and hence $\bar{E}^1_D(X) \cong \ker(\delta_0)$. □

Remark 4.2.4. If $D = \mathbb{C}$ or $D = \mathcal{Z}$ then A is automatically orientable since in those cases $K_0(D)_+^\times$ is the trivial group.

Remark 4.2.5. Let Y be a compact metrizable space and let $X = \Sigma Y$ be the suspension of Y. Since the rational Künneth isomorphism and the Chern character on $K^0(X)$ are compatible with the ring structure on $K_0(C(Y) \otimes D)$, we obtain a ring homomorphism

$$\mathrm{ch}\colon K_0(C(Y) \otimes D) \to K^0(Y) \otimes K_0(D) \otimes \mathbb{Q} \to \prod_{k=0}^{\infty} H^{2k}(Y, \mathbb{Q}) =: H^{\mathrm{ev}}(Y, \mathbb{Q}) \ ,$$

which restricts to a group homomorphism $\mathrm{ch}\colon \bar{E}^0_D(Y) \to SL_1(H^{\mathrm{ev}}(Y, \mathbb{Q}))$, where the right hand side denotes the units, which project to $1 \in H^0(Y, \mathbb{Q})$. If A is an orientable locally trivial continuous field with fiber $D \otimes \mathbb{K}$ over X, then we have

$$\delta_k(A) = \log \mathrm{ch}(f_A) \in H^{2k}(Y, \mathbb{Q}) \cong H^{2k+1}(X, \mathbb{Q}) \ , \tag{4.3}$$

where $f_A\colon Y \to \Omega B\mathrm{Aut}_0(D \otimes \mathbb{K}) \simeq \mathrm{Aut}_0(D \otimes \mathbb{K})$ is induced by the transition map of A. The homomorphism $\log\colon SL_1(H^{\mathrm{ev}}(Y, \mathbb{Q})) \to H^{\mathrm{ev}}(Y, \mathbb{Q})$ is the rational logarithm from [114, Section 2.5]. For the proof of (4.3) it suffices to treat the case $D = M_{\mathbb{Q}} \otimes \mathcal{O}_\infty$, where it can be easily checked on the level of homotopy groups, but since $\bar{E}^0_D(Y)$ and $H^{\mathrm{ev}}(Y, \mathbb{Q})$ have rational vector spaces as coefficients this is enough.

Lemma 4.2.6. *Let D be a strongly self-absorbing C^*-algebra in the class $\mathcal{D}$. If $p \in D \otimes \mathbb{K}$ is a projection such that $[p] \neq 0$ in $K_0(D)$, then there is an integer $n \geq 1$ such that $[p] \in nK_0(D)_+^\times$. If $[p] \in nK_0(D)_+^\times$, then $p(D \otimes \mathbb{K})p \cong M_n(D)$. Moreover, if $n, m \geq 1$, then $M_n(D) \cong M_m(D)$ if and only if $nK_0(D)_+^\times = mK_0(D)_+^\times$.*

Proof. Recall that $K_0(D)$ is an ordered unital ring with unit $[1_D]$ and with positive elements $K_0(D)_+$ corresponding to classes of projections in $D \otimes \mathbb{K}$. The group of invertible elements is denoted by $K_0(D)^\times$ and $K_0(D)_+^\times$ consists of classes $[p]$ of projections $p \in D \otimes \mathbb{K}$ such that $[p] \in K_0(D)^\times$. It was shown in [47, Lemma 2.14] that if $p \in D \otimes \mathbb{K}$ is a projection, then $[p] \in K_0(D)_+^\times$ if and only if $p(D \otimes \mathbb{K})p \cong D$. The ring $K_0(D)$ and the group $K_0(D)_+^\times$ are known for all $D \in \mathcal{D}$, [139]. In fact $K_0(D)$ is a unital subring of $\mathbb{Q}$, $K_0(D)_+ = \mathbb{Q}_+ \cap K_0(D)$ if $D \in \mathcal{D}_{qd}$ and $K_0(D)_+ = K_0(D)$ if $D \in \mathcal{D}_{pi}$. Moreover:

$$K_0(\mathbb{C}) \cong K_0(\mathcal{Z}) \cong K_0(\mathcal{O}_\infty) \cong \mathbb{Z},\ K_0(\mathcal{O}_2) = \{0\},$$

$$K_0(M_{P^\infty}) \cong K_0(M_{P^\infty} \otimes \mathcal{O}_\infty) \cong \mathbb{Z}[1/P] \cong \bigotimes_{p \in P} \mathbb{Z}[1/p] \cong \{np_1^{k_1}p_2^{k_2}\cdots p_r^{k_r} : p_i \in P, n, k_i \in \mathbb{Z}\},$$

$$K_0(\mathbb{C})_+^\times \cong K_0(\mathcal{Z})_+^\times = \{1\},\ K_0(\mathcal{O}_\infty)_+^\times = \{\pm 1\},$$

$$K_0(M_{P^\infty})_+^\times \cong \{p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r} : p_i \in P, k_i \in \mathbb{Z}\}.$$

$$K_0(M_{P^\infty} \otimes \mathcal{O}_\infty)_+^\times \cong \{\pm p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r} : p_i \in P, k_i \in \mathbb{Z}\}.$$

In particular, we see that in all cases $K_0(D)_+ = \mathbb{N} \cdot K_0(D)_+^\times$, which proves the first statement. If $p \in D \otimes \mathbb{K}$ is a projection such that $[p] \in nK_0(D)_+^\times$, then there is a projection $q \in D \otimes \mathbb{K}$ such that $[q] \in K_0(D)_+^\times$ and $[p] = n[q] = [\mathrm{diag}(q, q, \ldots, q)]$. Since D has cancellation of full projections, it follows then immediately that $p(D \otimes \mathbb{K})p \cong M_n(D)$ proving the second part.

To show the last part of the lemma, suppose now that $\alpha : D \otimes M_n(\mathbb{C}) \to D \otimes M_m(\mathbb{C})$ is a $*$-isomorphism. Let $e \in M_n(\mathbb{C})$ be a rank one projection. Then $\alpha(1_D \otimes e)(D \otimes M_m(\mathbb{C}))\alpha(1_D \otimes e) \cong D$. By [47, Lemma 2.14] it follows that $\alpha_*[1_D] = [\alpha(1_D \otimes e)] \in K_0(D)_+^\times$. Since α is unital, $\alpha_*(n[1_D]) = m[1_D]$ and hence $m[1_D] \in nK_0(D)_+^\times$. This is equivalent to $nK_0(D)_+^\times = mK_0(D)_+^\times$.

Conversely, suppose that $m[1_D] = nu$ for some $u \in K_0(D)_+^\times$. Let $\alpha \in \mathrm{Aut}(D \otimes \mathbb{K})$ be such that $[\alpha(1_D \otimes e)] = u$. Then $\alpha_*(n[1_D]) = nu = m[1_D]$. This implies that α maps a corner of $D \otimes \mathbb{K}$ that is isomorphic to $M_n(D)$ to a corner that is isomorphic to $M_m(D)$. □

Corollary 4.2.7. *Let $D \in \mathcal{D}$ and let $\theta \colon D \otimes M_{n^r}(\mathbb{C}) \to D \otimes M_{n^\infty}$ be a unital inclusion induced by some unital embedding $M_{n^r}(\mathbb{C}) \to M_{n^\infty}$, where $n \geq 2, r \geq 0$. Let R be the set of* *prime factors of n. Then, under the canonical isomorphism* $K_0(D \otimes M_{n^r}(\mathbb{C})) \cong K_0(D)$*, we have*

$$\theta_*^{-1}(K_0(D \otimes M_{n^\infty})_+^\times) = \bigcup_r rK_0(D)_+^\times \subset K_0(D)$$

where r *runs through the set of all products of the form* $\prod_{q \in R} q^{k_q}$*,* $k_q \in \mathbb{N} \cup \{0\}$*.*

Proof. From Lemma 4.2.6 we see that $K_0(D) \cong \mathbb{Z}[1/P]$ for a (possibly empty) set of primes P. The order structure is the one induced by $(\mathbb{Q}, \mathbb{Q}_+)$ if D is quasidiagonal or $K_0(D)^+ = \mathbb{Z}[1/P]$ if D is purely infinite. If $R \subseteq P$, then θ induces an isomorphism on K_0 and the statement is true, since θ_* is order preserving and $\mathbb{Z}[1/R]^\times \subseteq K_0(D)^\times$. Thus, we may assume that $R \not\subseteq P$. Let $S = P \cup R$ and thus $K_0(D \otimes M_{n^\infty}) \cong \mathbb{Z}[1/S]$. The map θ_* induces the canonical inclusion $\mathbb{Z}[1/P] \hookrightarrow \mathbb{Z}[1/S]$. We can write $x \in \mathbb{Z}[1/P]$ as

$$x = m \cdot \prod_{p \in P} p^{r_p} \cdot \prod_{q \in R \setminus P} q^{k_q}$$

with $m \in \mathbb{Z}$ relatively prime to all $p \in P$ and $q \in R$, only finitely many $r_p \in \mathbb{Z}$ non-zero and $k_q \in \mathbb{N} \cup \{0\}$. From this decomposition we see that x is invertible in $\mathbb{Z}[1/S]$ if and only if $m = \pm 1$. This concludes the proof since $p^{r_p} \in K_0(D)_+^\times$. $\square$

Remark 4.2.8. Let $q \in D \otimes \mathbb{K}$ be a projection and let $\alpha \in \mathrm{Aut}(D \otimes \mathbb{K})$. As in [47, Lemma 2.14] we have that $[\alpha(q)] = [\alpha(1 \otimes e)] \cdot [q]$ with $[\alpha(1 \otimes e)] \in K_0(D)_+^\times$. Thus, the condition $[q] \in nK_0(D)_+^\times$ for $n \in \mathbb{N}$ is invariant under the action of $\mathrm{Aut}(D \otimes \mathbb{K})$ on $K_0(D)$. Given $A \in \mathscr{C}_{D \otimes \mathbb{K}}(X)$, a projection $p \in A$, $x_0 \in X$ and an isomorphism $\phi\colon A(x_0) \to D \otimes \mathbb{K}$ the condition $[\phi(p(x_0))] \in nK_0(D)_+^\times$ is independent of ϕ. Abusing the notation we will write this as $[p(x_0)] \in nK_0(D)_+^\times$.

Corollary 4.2.9. *Let* $D \in \mathcal{D}$ *and let* $A \in \mathscr{C}_{D \otimes \mathbb{K}}(X)$ *with* X *a connected compact metrizable space. If* $p \in A$ *is a projection such that* $[p(x_0)] \in nK_0(D)_+^\times$ *for some point* x_0*, then* $(pAp)(x) \cong M_n(D)$ *for all* $x \in X$ *and hence* $pAp \in \mathscr{C}_{D \otimes M_n(\mathbb{C})}(X)$*. If* $p \in A$ *is a projection with* $[p(x_0)] \in K_0(D) \setminus \{0\}$*, then* $[p(x_0)] \in nK_0(D)_+^\times$ *for some* $n \in \mathbb{N}$*.*

Proof. Let $V_1, .., V_k$ be a finite cover of X by compact sets such that there are bundle isomorphisms $\phi_i : A(V_i) \cong C(V_i) \otimes D \otimes \mathbb{K}$. Let p_i be the image of the restriction of p to V_i under ϕ_i. After refining the cover (V_i), if necessary, we may assume that $\|p_i(x) - p_i(y)\| < 1$ for all $x, y \in V_i$. This allows us to find a unitary u_i in the multiplier algebra of $C(V_i) \otimes D \otimes \mathbb{K}$ such that after replacing ϕ_i by $u_i \phi_i u_i^*$ and p_i by $u_i p_i u_i^*$, we may assume that p_i are constant projections. Since X is connected and $[p(x_0)] \in nK_0(D)_+^\times$ by assumption, it follows from $[p_i(x_0)] \in nK_0(D)_+^\times$ for $x_0 \in V_i$ and the above remark that $[p_j(x)] \in nK_0(D)_+^\times$ for all $1 \leq j \leq k$ and all $x \in V_j$. Then Lemma 4.2.6 implies $(pAp)(V_j) \cong C(V_j) \otimes M_n(D)$.

By Lemma 4.2.6 we also have that $[p(x_0)] \neq 0$ implies $[p(x_0)] \in nK_0(D)_+^\times$ for some $n \in \mathbb{N}$ proving the statement about the case $[p(x_0)] \in K_0(D) \setminus \{0\}$. $\square$

We study the image of the stabilization map

$$\mathscr{C}_{D\otimes M_n(\mathbb{C})}(X) \to \mathscr{C}_{D\otimes\mathbb{K}}(X)$$

induced by the map $A \mapsto A \otimes \mathbb{K}$, or equivalently by the map

$$\mathrm{Aut}(D \otimes M_n(\mathbb{C})) \to \mathrm{Aut}(D \otimes M_n(\mathbb{C}) \otimes \mathbb{K}) \cong \mathrm{Aut}(D \otimes \mathbb{K}).$$

Let us recall that $\mathcal{D}$ denotes the class of strongly self-absorbing C^*-algebras which satisfy the UCT and which are either quasidiagonal or purely infinite.

Theorem 4.2.10. *Let D be a strongly self-absorbing C^*-algebra in the class $\mathcal{D}$. Let A be a locally trivial continuous field of C^*-algebras over a connected compact metrizable space X such that $A(x) \cong D \otimes \mathbb{K}$ for all $x \in X$. The following assertions are equivalent:*

(1) $\delta_k(A) = 0$ for all $k \geq 0$.

(2) The field $A \otimes M_{\mathbb{Q}}$ is trivial.

(3) There is an integer $n \geq 1$ and a unital locally trivial continuous field $\mathcal{B}$ over X with all fibers isomorphic to $M_n(D)$ such that $A \cong \mathcal{B} \otimes \mathbb{K}$.

(4) A is orientable and $A^{\otimes m} \cong C(X) \otimes D \otimes \mathbb{K}$ for some $m \in \mathbb{N}$.

Proof. The statement is immediately verified if $D \cong \mathcal{O}_2$. Indeed all locally trivial fields with fiber $\mathcal{O}_2 \otimes \mathbb{K}$ are trivial since $\mathrm{Aut}(\mathcal{O}_2 \otimes \mathbb{K})$ is contractible by [47, Cor. 17 & Thm. 2.17]. For the remainder of the proof we may therefore assume that $D \ncong \mathcal{O}_2$.

(1) $\Leftrightarrow$ (2) If $D \in \mathcal{D}_{qd}$, then it is known that $D \otimes M_{\mathbb{Q}} \cong M_{\mathbb{Q}}$. Similarly, if $D \in \mathcal{D}_{pi}$ and $D \ncong \mathcal{O}_2$ then $D \otimes M_{\mathbb{Q}} \cong \mathcal{O}_\infty \otimes M_{\mathbb{Q}}$. If A is as in the statement, then $A \otimes M_{\mathbb{Q}}$ is a locally trivial field whose fibers are all isomorphic to either $M_{\mathbb{Q}} \otimes \mathbb{K}$ or to $\mathcal{O}_\infty \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$. In either case, it was shown in [47, Cor. 4.5] that such a field is trivial if and only if $\delta_k(A) = 0$ for all $k \geq 0$. As reviewed earlier in this section, this follows from the explicit computation of $E^1_{M_{\mathbb{Q}}}(X)$ and $E^1_{M_{\mathbb{Q}}\otimes\mathcal{O}_\infty}(X)$.

(2) $\Rightarrow$ (3) Assume now that $A \otimes M_{\mathbb{Q}}$ is trivial, i.e. $A \otimes M_{\mathbb{Q}} \cong C(X) \otimes D \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$. Let $p \in A \otimes M_{\mathbb{Q}}$ be the projection that corresponds under this isomorphism to the projection $1 \otimes e \in C(X) \otimes D \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$ where 1 is the unit of the C^*-algebra $C(X) \otimes D \otimes M_{\mathbb{Q}}$ and $e \in \mathbb{K}$ is a rank-one projection. Then $[p(x)] \neq 0$ in $K_0(A(x) \otimes M_{\mathbb{Q}})$ for all $x \in X$ (recall that $D \ncong \mathcal{O}_2$). Let us write $M_{\mathbb{Q}}$ as the direct limit of an increasing sequence of its subalgebras $M_{k(i)}(\mathbb{C})$. Then $A \otimes M_{\mathbb{Q}}$ is the direct limit of the sequence $A_i = A \otimes M_{k(i)}(\mathbb{C})$. It follows that

there exist $i \geq 1$ and a projection $p_i \in A_i$ such that $\|p - p_i\| < 1$. Then $\|p(x) - p_i(x)\| < 1$ and so $[p_i(x)] \neq 0$ in $K_0(A_i(x))$ for each $x \in X$, since its image in $K_0(A(x) \otimes M_{\mathbb{Q}})$ is equal to $[p(x)] \neq 0$. Let us consider the locally trivial unital field $\mathcal{B} := p_i(A \otimes M_{k(i)}(\mathbb{C}))p_i$. Since the fibers of $A \otimes M_{k(i)}(\mathbb{C})$ are isomorphic to $D \otimes \mathbb{K} \otimes M_{k(i)}(\mathbb{C}) \cong D \otimes \mathbb{K}$, it follows by Corollary 4.2.9 that there is $n \geq 1$ such that all fibers of $\mathcal{B}$ are isomorphic to $M_n(D)$. Since $\mathcal{B}$ is isomorphic to a full corner of $A \otimes \mathbb{K}$, it follows by [24] that $A \otimes \mathbb{K} \cong \mathcal{B} \otimes \mathbb{K}$. We conclude by noting that since A is locally trivial and each fiber is stable, then $A \cong A \otimes \mathbb{K}$ by [72] and so $A \cong \mathcal{B} \otimes \mathbb{K}$.

(3) $\Rightarrow$ (2) This implication holds for any strongly self-absorbing C^*-algebra D. Let A and $\mathcal{B}$ be as in (3). Let us note that $\mathcal{B} \otimes M_{\mathbb{Q}}$ is a unital locally trivial field with all fibers isomorphic to the strongly self-absorbing C^*-algebra $D \otimes M_{\mathbb{Q}}$. Since $\mathrm{Aut}(D \otimes M_{\mathbb{Q}})$ is contractible by [47, Thm. 2.3], it follows that $\mathcal{B} \otimes M_{\mathbb{Q}}$ is trivial. We conclude that $A \otimes M_{\mathbb{Q}} \cong (\mathcal{B} \otimes M_{\mathbb{Q}}) \otimes \mathbb{K} \cong C(X) \otimes D \otimes M_{\mathbb{Q}} \otimes \mathbb{K}$.

(2) $\Leftrightarrow$ (4) This equivalence holds for any strongly self-absorbing C^*-algebra D if A is orientable. In particular we do not need to assume that D satisfies the UCT. In the UCT case we note that since the map $K_0(D) \to K_0(D \otimes M_{\mathbb{Q}})$ is injective, it follows that A is orientable if and only if $A \otimes M_{\mathbb{Q}}$ is orientable, i.e. $\delta_0(A) = 0$ if and only if $\delta_0^s(A) = 0$. Since $\delta_0(A) = 0$, A is determined up to isomorphism by its class $[A] \in \bar{E}^1_D(X)$. To complete the proof it suffices to show that the kernel of the map $\tau : \bar{E}^1_D(X) \to \bar{E}^1_{D \otimes M_{\mathbb{Q}}}(X)$, $\tau[A] = [A \otimes M_{\mathbb{Q}}]$, consists entirely of torsion elements. Consider the natural transformation of cohomology theories:

$$\tau \otimes \mathrm{id}_{\mathbb{Q}} : \bar{E}^*_D(X) \otimes \mathbb{Q} \to \bar{E}^*_{D \otimes M_{\mathbb{Q}}}(X) \otimes \mathbb{Q} \cong \bar{E}^*_{D \otimes M_{\mathbb{Q}}}(X).$$

If $D \neq \mathbb{C}$, it induces an isomorphism on coefficients since $\bar{E}^{-i}_D(pt) = \pi_i(\mathrm{Aut}_0(D \otimes \mathbb{K})) \cong K_i(D)$ by [47, Thm.2.18] and since the map $K_i(D) \otimes \mathbb{Q} \to K_i(D \otimes M_{\mathbb{Q}})$ is bijective. We conclude that the kernel of τ is a torsion group. The same property holds for $D = \mathbb{C}$ since $\bar{E}^*_{\mathbb{C}}(X)$ is a direct summand of $\bar{E}^*_{\mathcal{Z}}(X)$ by [47, Cor.3.8]. $\square$

Theorem 4.2.11. *Let D, X and A be as in Theorem 4.2.10 and let $n \geq 2$ be an integer. The following assertions are equivalent:*

(1) The field $A \otimes M_{n^\infty}$ is trivial.

(2) There is a $k \in \mathbb{N}$ and a unital locally trivial continuous field $\mathcal{B}$ over X with all fibers isomorphic to $M_{n^k}(D)$ such that $A \cong \mathcal{B} \otimes \mathbb{K}$.

(3) A is orientable and $A^{\otimes n^k} \cong C(X) \otimes D \otimes \mathbb{K}$ for some $k \in \mathbb{N}$.

Proof. By reasoning as in the proof of Theorem 4.2.10, we may assume that $D \not\cong \mathcal{O}_2$.

(1) $\Rightarrow$ (2): By assumption the continuous field $A \otimes M_{n^\infty}$ is trivializable and hence it satisfies the global Fell condition of [47]. This means that there is a full projection

$p_\infty \in A \otimes M_{n^\infty}$ with the property that $p_\infty(x) \in K_0(A(x) \otimes M_{n^\infty})_+^\times$ for all $x \in X$. Let $\nu_i \colon M_{n^i}(\mathbb{C}) \to M_{n^\infty}$ be a unital inclusion map. Since $A \otimes M_{n^\infty}$ is the inductive limit of the sequence

$$A \to A \otimes M_n(\mathbb{C}) \to \cdots \to A \otimes M_{n^i}(\mathbb{C}) \to A \otimes M_{n^{i+1}}(\mathbb{C}) \to \dots$$

there is an $i \in \mathbb{N}$ and a full projection $p \in A \otimes M_{n^i}(\mathbb{C})$ with $\|(\mathrm{id}_A \otimes \nu_i)(p) - p_\infty\| < 1$. Fix a point $x_0 \in X$. Let $\theta \colon A(x_0) \otimes M_{n^i}(\mathbb{C}) \to A(x_0) \otimes M_{n^\infty}$ be the unital inclusion induced by ν_i. Note that $\theta_*([p(x_0)]) = (\mathrm{id}_{A(x_0)} \otimes \nu_i)_*([p(x_0)]) = [p_\infty(x_0)] \in K_0(A(x_0) \otimes M_{n^\infty})_+^\times$. By Corollary 4.2.7 this implies that $[p(x_0)] \in rK_0(A(x_0))_+^\times$ for some $r \in \mathbb{N}$ that divides n^k for some $k \in \mathbb{N} \cup \{0\}$. Then $\mathcal{B}_0 := p(A \otimes M_{n^i}(\mathbb{C}))p \in \mathscr{C}_{D \otimes M_r(\mathbb{C})}(X)$ by Corollary 4.2.9. Write $n^k = mr$ with $m \in \mathbb{N}$. It follows that $\mathcal{B} := \mathcal{B}_0 \otimes M_m(\mathbb{C}) \in \mathscr{C}_{D \otimes M_{n^k}(\mathbb{C})}(X)$. The fact that $\mathcal{B} \otimes \mathbb{K} \cong A$ follows just as in step (2) $\Rightarrow$ (3) in the proof of Theorem 4.2.10.

(2) $\Rightarrow$ (1): This is just the same argument as step (3) $\Rightarrow$ (2) in the proof of Theorem 4.2.10.

(1) $\Leftrightarrow$ (3): The orientability of A follows from Theorem 4.2.10. Observe that the elements $[A] \in \mathscr{C}^0_{D \otimes \mathbb{K}}(X) = \bar{E}^1_D(X)$ such that $n^k[A] = 0$ or equivalently $A^{\otimes n^k}$ is trivializable for some $k \in \mathbb{N} \cup \{0\}$ coincide precisely with the elements in the kernel of the group homomorphism $\bar{E}^1_D(X) \to \bar{E}^1_D(X) \otimes \mathbb{Z}[\frac{1}{n}]$. Since $\mathbb{Z}[\frac{1}{n}]$ is flat, it follows that $X \mapsto \bar{E}^*_D(X) \otimes \mathbb{Z}[\frac{1}{n}]$ still satisfies all axioms of a generalized cohomology theory. In particular, we have the following commutative diagram of natural transformations of cohomology theories:

$$\begin{tikzcd}
\bar{E}^*_D(X) \arrow[r] \arrow[d] & \bar{E}^*_{D \otimes M_{n^\infty}}(X) \arrow[d, "\cong"] \\
\bar{E}^*_D(X) \otimes \mathbb{Z}[\frac{1}{n}] \arrow[r] & \bar{E}^*_{D \otimes M_{n^\infty}}(X) \otimes \mathbb{Z}[\frac{1}{n}]
\end{tikzcd}$$

where the isomorphism on the right hand side can be checked on the coefficients. A similar argument shows that for $D \neq \mathbb{C}$ the bottom homomorphism is an isomorphism. Thus the kernel of the left vertical map agrees with the one of the upper horizontal map in this case. For $D = \mathbb{C}$ we can use that $\bar{E}^*_{\mathbb{C}}(X)$ embeds as a direct summand into $\bar{E}^*_{\mathcal{Z}}(X)$ via the natural $*$-homomorphism $\mathbb{C} \to \mathcal{Z}$ [47, Cor. 4.8]. In particular, $\bar{E}^*_{\mathbb{C}}(X) \otimes \mathbb{Z}[\frac{1}{n}] \to \bar{E}^*_{\mathcal{Z}}(X) \otimes \mathbb{Z}[\frac{1}{n}]$ is injective. □

Corollary 4.2.12. *Let D and X be as in Theorem 4.2.10. Then any element $x \in \bar{E}^1_D(X)$ with $nx = 0$ is represented by the stabilization of a unital locally trivial field over X with all fibers isomorphic to $M_{n^k}(D)$ for some $k \geq 1$. Moreover if $A \in \mathscr{C}_{D \otimes \mathbb{K}}(X)$, then $A \otimes M_{\mathbb{Q}}$ is trivial $\Leftrightarrow$ $A \otimes M_{n^\infty}$ is trivial for some $n \in \mathbb{N}$ $\Leftrightarrow$ A is orientable and $n^k[A] = 0$ in $\bar{E}^1_D(X)$ for some $k \in \mathbb{N}$ and some $n \in \mathbb{N}$.*

(An example from [8] for $D = \mathbb{C}$ shows that in general one cannot always arrange that

$k = 1$.)

Proof. The first part follows from Theorem 4.2.11. Indeed, condition (3) of that theorem is equivalent to requiring that A is orientable and $n^k[A] = 0$ in $\bar{E}^1_D(X)$. The second part follows from Theorems 4.2.10 and 4.2.11. □

Definition 4.2.13. Let D be a strongly self-absorbing C^*-algebra. If X is connected compact metrizable space we define the Brauer group $Br_D(X)$ as equivalence classes of continuous fields $A \in \bigcup_{n\geq 1} \mathscr{C}_{M_n(D)}(X)$. Two continuous fields $A_i \in \mathscr{C}_{M_{n_i}(D)}(X)$, $i = 1, 2$ are equivalent, if

$$A_1 \otimes p_1 C(X, M_{N_1}(D))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(D))p_2,$$

for some full projections $p_i \in C(X, M_{N_i}(D))$. We denote by $[A]_{Br}$ the class of A in $Br_D(X)$. The multiplication on $Br_D(X)$ is induced by the tensor product operation, after fixing an isomorphism $D \otimes D \cong D$. We will show in a moment that the monoid $Br_D(X)$ is a group.

Remark 4.2.14. It is worth noting the following two alternative descriptions of the Brauer group. (a) If $D \in \mathcal{D}$ is quasidiagonal, then two continuous fields $A_i \in \mathscr{C}_{M_{n_i}(D)}(X)$, $i = 1, 2$ have equal classes in $Br_D(X)$, if and only if $A_1 \otimes p_1 C(X, M_{N_1}(\mathbb{C}))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathbb{C}))p_2$, for some full projections $p_i \in C(X, M_{N_i}(\mathbb{C}))$. (b) If $D \in \mathcal{D}$ is purely infinite, then two continuous fields $A_i \in \mathscr{C}_{M_{n_i}(D)}(X)$, $i = 1, 2$ have equal classes in $Br_D(X)$, if and only if $A_1 \otimes p_1 C(X, M_{N_1}(\mathcal{O}_\infty))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathcal{O}_\infty))p_2$, for some full projections $p_i \in C(X, M_{N_i}(\mathcal{O}_\infty))$. In order to justify (a) we observe that if D is quasidiagonal, then every projection $p \in C(X, M_N(D))$ has a multiple $p(m) := p \otimes 1_{M_m}(\mathbb{C})$ such that $p(m)$ is Murray-Von Neumann equivalent to a projection in $C(X, M_{Nm}(\mathbb{C})) \otimes 1_D \subset C(X, M_{Nm}(\mathbb{C})) \otimes D$ and that $A_i \otimes D \cong A_i$ by [72]. For (b) we note that if D is purely infinite, then then every projection $p \in C(X, M_N(D))$ has a multiple $p \otimes 1_{M_m}(\mathbb{C})$ that is Murray-Von Neumann equivalent to a projection in $C(X, M_{Nm}(\mathcal{O}_\infty)) \otimes 1_D$.

One has the following generalization of a result of Serre, [67, Thm.1.6].

Theorem 4.2.15. *Let D be a strongly self-absorbing C^*-algebra in $\mathcal{D}$.*

(i) $Tor(\bar{E}^1_D(X)) = ker\left(\bar{E}^1_D(X) \xrightarrow{\bar{\delta}} \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q})\right)$

(ii) The map $\theta : Br_D(X) \to Tor(\bar{E}^1_D(X))$, $[A]_{Br} \mapsto [A \otimes \mathbb{K}]$ *is an isomorphism of groups.*

Proof. (i) was established in the last part of the proof of Theorem 4.2.10.

(ii) We denote by L_p the continuous field $p\,C(X, M_N(D))p$. Since $L_p \otimes \mathbb{K} \cong C(X, D \otimes \mathbb{K})$ it follows that the map θ is a well-defined morphism of monoids.

We use the following observation. Let $\theta : S \to G$ be a unital surjective morphism of commutative monoids with units denoted by 1. Suppose that G is a group and that

$\{s \in S\colon \theta(s) = 1\} = \{1\}$. Then S is a group and θ is an isomorphism. Indeed if $s \in S$, there is $t \in S$ such that $\theta(t) = \theta(s)^{-1}$ by surjectivity of θ. Then $\theta(st) = \theta(s)\theta(t) = 1$ and so $st = 1$. It follows that S is a group and that θ is injective.

We are going to apply this observation to the map $\theta : Br_D(X) \to Tor(\bar{E}^1_D(X))$. By condition (3) of Theorem 4.2.10 we see that θ is surjective. Let us determine the set $\theta^{-1}(\{0\})$. We are going to show that if $B \in \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X)$, then $[B \otimes \mathbb{K}] = 0$ in $\bar{E}^1_D(X)$ if and only if

$$B \cong p\,(C(X) \otimes D \otimes M_N(\mathbb{C}))\,p \cong \mathcal{L}_{C(X,D)}(p\,C(X,D)^N)$$

for some selfadjoint projection $p \in C(X) \otimes D \otimes M_N(\mathbb{C}) \cong M_N(C(X,D))$. Let $B \in \mathscr{C}_{D\otimes M_n(\mathbb{C})}(X)$ be such that $[B \otimes \mathbb{K}] = 0$ in $\bar{E}^1_D(X)$. Then there is an isomorphism of continuous fields $\phi : B\otimes\mathbb{K} \xrightarrow{\cong} C(X)\otimes D\otimes\mathbb{K}$. After conjugating ϕ by a unitary we may assume that $p := \phi(1_B \otimes e_{11}) \in C(X) \otimes D \otimes M_N(\mathbb{C})$ for some integer $N \geq 1$. It follows immediately that the projection p has the desired properties. Conversely, if $B \cong p\,(C(X) \otimes D \otimes M_N(\mathbb{C}))\,p$ then there is an isomorphism of continuous fields $B \otimes \mathbb{K} \cong C(X) \otimes D \otimes \mathbb{K}$ by [24]. We have thus shown that that $\theta([B]_{Br}) = 0$ iff and only if $[B]_{Br} = 0$.

We are now able to conclude that $Br_D(X)$ is a group and that θ is injective by the general observation made earlier. □

Definition 4.2.16. Let D be a strongly self-absorbing C^*-algebra. Let A be a locally trivial continuous field of C^*-algebras with fiber $D\otimes\mathbb{K}$. We say that A is a *torsion continuous field* if $A^{\otimes k}$ is isomorphic to a trivial field for some integer $k \geq 1$.

Corollary 4.2.17. *Let A be as in Theorem 4.2.10. Then A is a torsion continuous field if and only if $\delta_0(A) \in H^1(X, K_0(D)^\times_+)$ is a torsion element and $\delta_k(A) = 0 \in H^{2k+1}(X,\mathbb{Q})$ for all $k \geq 1$.*

Proof. Let $m \geq 1$ be an integer such that $m\delta_0(A) = 0$. Then $\delta_0(A^{\otimes m}) = 0$. We conclude by applying Theorem 4.2.10 to the orientable continuous field $A^{\otimes m}$. □

4.3 Characteristic classes of the opposite continuous field

Given a C^*-algebra B denote by B^{op} the *opposite C^*-algebra* with the same underlying Banach space and norm, but with multiplication given by $b^{\mathrm{op}} \cdot a^{\mathrm{op}} = (a \cdot b)^{\mathrm{op}}$. The *conjugate C^*-algebra* $\overline{B}$ has the conjugate Banach space as its underlying vector space, but the same multiplicative structure. The map $a \mapsto a^*$ provides an isomorphism $B^{\mathrm{op}} \to \overline{B}$. Any automorphism $\alpha \in \mathrm{Aut}(B)$ yields in a canonical way automorphisms $\bar{\alpha}\colon \overline{B} \to \overline{B}$ and $\alpha^{\mathrm{op}}\colon B^{\mathrm{op}} \to B^{\mathrm{op}}$ compatible with $*\colon B^{\mathrm{op}} \to \overline{B}$. Therefore we have group isomorphisms $\theta\colon \mathrm{Aut}(B) \to \mathrm{Aut}(\overline{B})$ and $\mathrm{Aut}(B) \to \mathrm{Aut}(B^{\mathrm{op}})$. Note that $\alpha \in \mathrm{Aut}(B)$ is equal to $\theta(\alpha)$ when regarded as set-theoretic maps $B \to B$. Given a locally trivial continuous field A with fiber B, we can apply these operations fiberwise to obtain the locally trivial fields A^{op} and $\overline{A}$, which we will call the *opposite* and the *conjugate field*. They are isomorphic to each other and isomorphic to the conjugate and the opposite C^*-algebras of A.

A *real form* of a complex C*-algebra A is a real C*-algebra $A^{\mathbb{R}}$ such that $A \cong A^{\mathbb{R}} \otimes \mathbb{C}$. A real form is not necessarily unique [20] and not all C*-algebras admit real forms [112]. If two C*-algebras A and B admit real forms $A^{\mathbb{R}}$ and $B^{\mathbb{R}}$, then $A^{\mathbb{R}} \otimes_{\mathbb{R}} B^{\mathbb{R}}$ is a real form of $A \otimes B$.

Example 4.3.1. All known strongly self-absorbing C*-algebras $D \in \mathcal{D}$ admit a real form.

Indeed, the real Cuntz algebras $\mathcal{O}_2^{\mathbb{R}}$ and $\mathcal{O}_\infty^{\mathbb{R}}$ are defined by the same generators and relations as their complex versions. Alternatively $\mathcal{O}_\infty^{\mathbb{R}}$ can be realized as follows. Let $H_{\mathbb{R}}$ be a separable infinite dimensional real Hilbert space and let $\mathcal{F}^{\mathbb{R}}(H_{\mathbb{R}}) = \bigoplus_{n=0}^{\infty} H_{\mathbb{R}}^{\otimes n}$ be the real Fock space associated to it. Every $\xi \in H_{\mathbb{R}}$ defines a shift operator $s_\xi(\eta) = \xi \otimes \eta$ and we denote the algebra spanned by the s_ξ and their adjoints s_ξ^* by $\mathcal{O}_\infty^{\mathbb{R}}$. If $\mathcal{F}(H_{\mathbb{R}} \otimes \mathbb{C})$ denotes the Fock space associated to the complex Hilbert space $H = H_{\mathbb{R}} \otimes \mathbb{C}$, then we have $\mathcal{F}^{\mathbb{R}} \otimes \mathbb{C} \cong \mathcal{F}(H)$. If we represent $\mathcal{O}_\infty$ on $\mathcal{F}(H)$ using the above construction, then the map $s_\xi + i\, s_{\xi'} \mapsto s_{\xi + i\,\xi'}$ induces an isomorphism $\mathcal{O}_\infty^{\mathbb{R}} \otimes \mathbb{C} \to \mathcal{O}_\infty$. Likewise define $M_{\mathbb{Q}}^{\mathbb{R}}$ to be the infinite tensor product $M_2(\mathbb{R}) \otimes M_3(\mathbb{R}) \otimes M_4(\mathbb{R}) \otimes \dots$. Since $M_n(\mathbb{C}) \cong M_n(\mathbb{R}) \otimes \mathbb{C}$, we obtain an isomorphism $M_{\mathbb{Q}}^{\mathbb{R}} \otimes \mathbb{C} \cong M_{\mathbb{Q}}$ on the inductive limit. Let $\mathbb{K}^{\mathbb{R}}$ be the compact operators on $H_{\mathbb{R}}$ and $\mathbb{K}$ those on H, then we have $\mathbb{K}^{\mathbb{R}} \otimes \mathbb{C} \cong \mathbb{K}$. Thus, $M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ is the complexification of the real C^*-algebra $M_{\mathbb{Q}}^{\mathbb{R}} \otimes \mathcal{O}_\infty^{\mathbb{R}} \otimes \mathbb{K}^{\mathbb{R}}$.

The Jiang-Su algebra $\mathcal{Z}$ admits a real form $\mathcal{Z}^{\mathbb{R}}$ which can be constructed in the same way as $\mathcal{Z}$. Indeed, one constructs $\mathcal{Z}^{\mathbb{R}}$ as the inductive limit of a system

$$\cdots \to C([0,1], M_{p_n q_n}(\mathbb{R})) \xrightarrow{\phi_n} C([0,1], M_{p_{n+1} q_{n+1}}(\mathbb{R})) \to \cdots$$

where the connecting maps ϕ_n are defined just as in the proof of [78, Prop. 2.5] with only one modification. Specifically, one can choose the matrices u_0 and u_1 to be in the special orthogonal group $SO(p_n q_n)$ and this will ensure the existence of a continuous path u_t in $O(p_n q_n)$ from u_0 to u_1 as required.

If B is the complexification of a real C^*-algebra $B^{\mathbb{R}}$, then a choice of isomorphism $B \cong B^{\mathbb{R}} \otimes \mathbb{C}$ provides an isomorphism $c\colon B \to \overline{B}$ via complex conjugation on $\mathbb{C}$. On automorphisms we have $\mathrm{Ad}_{c^{-1}}\colon \mathrm{Aut}(\overline{B}) \to \mathrm{Aut}(B)$. Let $\eta = \mathrm{Ad}_{c^{-1}} \circ \theta\colon \mathrm{Aut}(B) \to \mathrm{Aut}(B)$. Now we specialize to the case $B = D \otimes \mathbb{K}$ with $D \in \mathcal{D}$ and study the effect of η on homotopy groups, i.e. $\eta_*\colon \pi_{2k}(\mathrm{Aut}(B)) \to \pi_{2k}(\mathrm{Aut}(B))$. By [47, Theorem 2.18] the groups $\pi_{2k+1}(\mathrm{Aut}(B))$ vanish.

Let R be a commutative ring and denote by $\left[K^0(S^{2k}) \otimes R\right]^\times$ the group of units of the ring $K^0(S^{2k}) \otimes R$. Let $\left[K^0(S^{2k}) \otimes R\right]_1^\times$ be the kernel of the morphism of multiplicative groups $\left[K^0(S^{2k}) \otimes R\right]^{\times} \to R^{\times}$. This is the group of virtual rank 1 vector bundles with coefficients in R over S^{2k}. Let $c_S \colon K^0(S^{2k}) \to K^0(S^{2k})$ and $c_R \colon K_0(D) \to K_0(D)$ be the ring automorphisms induced by complex conjugation.

Lemma 4.3.2. *Let D be a strongly self-absorbing C^*-algebra in the class $\mathcal{D}$, let $R = K_0(D)$ and let $k > 0$. There is an isomorphism $\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) \to \left[K^0(S^{2k}) \otimes R\right]^{\times}_1$ ($k > 0$) such that the following diagram commutes*

$$\begin{tikzcd}
\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) \arrow[r, "\eta_*"] \arrow[d] & \pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) \arrow[d] \\
\left[K^0(S^{2k}) \otimes R\right]^{\times}_1 \arrow[r, "c_S \otimes c_R"] & \left[K^0(S^{2k}) \otimes R\right]^{\times}_1
\end{tikzcd}$$

Proof. Observe that $\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) = \pi_{2k}(\mathrm{Aut}_0(D \otimes \mathbb{K}))$ (for $k > 0$) and $\mathrm{Aut}_0(D \otimes \mathbb{K})$ is a path connected group, therefore $\pi_{2k}(\mathrm{Aut}(D \otimes \mathbb{K})) = [S^{2k}, \mathrm{Aut}_0(D \otimes \mathbb{K})]$. Let $e \in \mathbb{K}$ be a rank 1 projection such that $c(1_D \otimes e) = 1_D \otimes e$. It follows from the proof of [47, Theorem 2.22] that the map $\alpha \mapsto \alpha(1 \otimes e)$ induces an isomorphism $[S^{2k}, \mathrm{Aut}_0(D \otimes \mathbb{K})] \to K_0(C(S^{2k}) \otimes D)^{\times}_1 = 1 + K_0(C_0(S^{2k} \setminus x_0) \otimes D)$. We have $\eta(\alpha)(1 \otimes e) = c^{-1}(\alpha(c(1 \otimes e))) = c^{-1}(\alpha(1 \otimes e))$, i.e. the isomorphism intertwines η and c^{-1}. Consider the following diagram of rings:

$$\begin{tikzcd}
K^0(S^{2k}) \otimes R \arrow[r, "c_S \otimes c_R"] \arrow[d] & K^0(S^{2k}) \otimes R \arrow[d] \\
K_0(C(S^{2k}) \otimes D) \arrow[r, "p \mapsto c^{-1}(p)"] & K_0(C(S^{2k}) \otimes D)
\end{tikzcd}$$

The vertical maps arise from the Künneth theorem. Since $K_1(D) = 0$, these are isomorphisms. Since c_S corresponds to the operation induced on $K_0(C(S^{2k}))$ by complex conjugation on $\mathbb{K}$, the above diagram commutes. $\square$

Remark 4.3.3. (i) If $D \in \mathcal{D}$ then $R = K_0(D) \subset \mathbb{Q}$ with $[1_D] = [1_{D^{\mathbb{R}}}] = 1$. Thus $c^{-1}(1_D) = 1_D$ and this shows that the above automorphism c_R is trivial. The K^0-ring of the sphere is given by $K^0(S^{2k}) \cong \mathbb{Z}[X_k]/(X_k^2)$. The element X_k is the k-fold reduced exterior tensor power of $H - 1$, where H is the tautological line bundle over $S^2 \cong \mathbb{C}P^1$. Since c_S maps $H - 1$ to $1 - H$, it follows that X_k is mapped to $-X_k$ if k is odd and to X_k if k is even. We have $[K^0(S^2) \otimes R]^{\times}_1 = \{1 + t\, X_k \mid t \in R\} \subset R[X_k]/(X_k^2)$. Thus, c_S maps $1 + t\, X_k$ to its inverse $1 - t\, X_k$ if k is odd and acts trivially if k is even.

(ii) By [47, Theorem 2.18] there is an isomorphism $\pi_0(\mathrm{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^{\times}_+$ given by $[\alpha] \mapsto [\alpha(1 \otimes e)]$. Arguing as in Lemma 4.3.2 we see that the action of η on this groups is given by $c_R = \mathrm{id}$.

Theorem 4.3.4. *Let X be a compact metrizable space and let A be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ for a strongly self-absorbing C^*-algebra $D \in \mathcal{D}$. Then we have for $k \geq 0$:*

$$\delta_k(A^{\mathrm{op}}) = \delta_k(\overline{A}) = (-1)^k\, \delta_k(A) \in H^{2k+1}(X, \mathbb{Q}) \ .$$

Proof. Let $D^{\mathbb{R}}$ be a real form of D. The group isomorphism $\eta\colon \mathrm{Aut}(D \otimes \mathbb{K}) \to \mathrm{Aut}(D \otimes \mathbb{K})$ induces an infinite loop map $B\eta\colon B\mathrm{Aut}(D \otimes \mathbb{K}) \to B\mathrm{Aut}(D \otimes \mathbb{K})$, where the infinite loop space structure is the one described in [47, Section 3]. If $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ is the classifying map of a locally trivial field A, then $B\eta \circ f$ classifies $\overline{A}$. Thus the induced map $\eta_* : E^1_D(X) \to E^1_D(X)$ has the property that $\eta_*[A] = [\overline{A}]$.

The unital inclusion $D^{\mathbb{R}} \to B^{\mathbb{R}} := D^{\mathbb{R}} \otimes \mathcal{O}^{\mathbb{R}}_\infty \otimes M^{\mathbb{R}}_{\mathbb{Q}}$ induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}(D \otimes \mathbb{K}) \arrow[r, "\eta"] \arrow[d] & \mathrm{Aut}(D \otimes \mathbb{K}) \arrow[d] \\
\mathrm{Aut}(B \otimes \mathbb{K}) \arrow[r, "\eta"] & \mathrm{Aut}(B \otimes \mathbb{K})
\end{tikzcd}$$

with $B := B^{\mathbb{R}} \otimes \mathbb{C}$. From this we obtain a commutative diagram

$$\begin{tikzcd}
E^1_D(X) \arrow[r, "\eta_*"] \arrow[d, "\delta"'] & E^1_D(X) \arrow[d, "\delta"] \\
E^1_B(X) \arrow[r, "\eta_*"] & E^1_B(X)
\end{tikzcd}$$

As explained earlier, $B \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty$. Recall that

$$E^1_{M_{\mathbb{Q}} \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \ .$$

By Lemma 4.3.2 and Remark 4.3.3(i) the effect of η on $H^{2k+1}(X, \pi_{2k}(\mathrm{Aut}(B))) \cong H^{2k+1}(X, \mathbb{Q})$ is given by multiplication with $(-1)^k$ for $k > 0$. By Remark 4.3.3(ii) η acts trivially on $H^1(X, \pi_0(\mathrm{Aut}(B))) = H^1(X, \mathbb{Q}^\times)$. □

Example 4.3.5. Let $\mathcal{Z}$ be the Jiang-Su algebra. We will show that in general the inverse of an element in the Brauer group $Br_{\mathcal{Z}}(X)$ is not represented by the class of the opposite algebra. Let Y be the space obtained by attaching a disk to a circle by a degree three map and let $X_n = S^n \wedge Y$ be n^{th} reduced suspension of Y. Then $E^1_{\mathcal{Z}}(X_3) \cong K^0(X_2)^\times_+ \cong 1 + \widetilde{K}^0(X_2)$ by [47, Thm.2.22]. Since this is a torsion group, $Br_{\mathcal{Z}}(X_3) \cong E^1_{\mathcal{Z}}(X_3)$ by Theorem 4.2.15. Using the Künneth formula, $Br_{\mathcal{Z}}(X_3) \cong 1 + \widetilde{K}^0(S^2) \otimes \widetilde{K}^0(Y) \cong 1 + \mathbb{Z}/3$. Reasoning as in Lemma 4.3.2 with X_2 in place of S^{2k}, we identify the map $\eta_* : E^1_{\mathcal{Z}}(X_3) \to E^1_{\mathcal{Z}}(X_3)$ with the map $K^0(X_2)^\times_+ \to K^0(X_2)^\times_+$ that sends the class $x = [V_1] - [V_2]$ to $\overline{x} = [\overline{V}_1] - [\overline{V}_2]$, where $\overline{V}_i$ is the complex conjugate bundle of V_i. If V is complex vector bundle, and c_1 is the first Chern class, $c_1(\overline{V}) = -c_1(V)$ by [81, p.206]. Since conjugation is compatible with the Künneth formula, we deduce that $x = \overline{x}$ for $x \in K^0(X_2)_+^\times$. Indeed, if $\beta \in \widetilde{K}^0(S^2)$, $y \in \widetilde{K}^0(Y)$ and $x = 1 + \beta y$, then $\overline{x} = 1 + (-\beta)(-y) = x$. Let A be a continuous field over X_3 with fibers $M_N(\mathcal{Z})$ such that $[A]_{Br} = 1 + \beta y$ in $Br_{\mathcal{Z}}(X_3) \cong 1 + \widetilde{K}^0(S^2) \otimes \widetilde{K}^0(Y) \cong 1 + \mathbb{Z}/3$, where β a generator of $\widetilde{K}^0(S^2)$ and y is a generator of $\widetilde{K}^0(Y)$. Then $[\overline{A}]_{Br} = 1 + (-\beta)(-y) = [A]_{Br}$ and hence

$$[\overline{A} \otimes_{C(X_3)} A]_{Br} = (1 + \beta y)^2 = 1 + 2\beta y \neq 1.$$

Corollary 4.3.6. *Let X be a compact metrizable space and let A be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ with D in the class $\mathcal{D}$. If $H^{4k+1}(X, \mathbb{Q}) = 0$ for all $k \geq 0$, then there is an $N \in \mathbb{N}$ such that*

$$(A \otimes_{C(X)} A^{\mathrm{op}})^{\otimes N} \cong C(X, D \otimes \mathbb{K}) \ .$$

Proof. If $H^{4k+1}(X, \mathbb{Q}) = 0$, then $\delta_{2k}(A \otimes_{C(X)} A^{\mathrm{op}}) = 0$ for all $k \geq 0$. Moreover, $\delta_{2k+1}(A \otimes_{C(X)} A^{\mathrm{op}}) = \delta_{2k+1}(A) - \delta_{2k+1}(A) = 0$. The statement follows from Corollary 4.2.17. □

5 A noncommutative model for higher twisted K-theory

We develop an operator algebraic model for twisted K-theory, which includes the most general twistings as a generalized cohomology theory (i.e. all those classified by the unit spectrum $bgl_1(KU)$). Our model is based on strongly self-absorbing C^*-algebras. We compare it with the known homotopy theoretic descriptions in the literature, which either use parametrized stable homotopy theory or ∞-categories. We derive a similar comparison of analytic twisted K-homology with its topological counterpart based on generalized Thom spectra. Our model also works for twisted versions of localizations of the K-theory spectrum, like $KU[1/n]$ or $KU_{\mathbb{Q}}$.

5.1 Introduction

It was observed by Donovan and Karoubi in [56] that there is a Thom isomorphism (and Poincaré duality) in the absence of K-orientability if one considers K-theory with local coefficients, also known as twisted K-theory. Whereas ordinary operator algebraic K-theory involves matrix-valued functions on a space, twisted K-theory replaces those by sections of bundles of matrix algebras. Morita equivalence classes of the latter are classified by torsion elements in $H^3(X, \mathbb{Z})$. Rosenberg [122] and Atiyah and Segal [8] extended the definition in [56] by considering bundles of compact operators corresponding to non-torsion twists. The observation that twisted K-groups classify D-brane charges sparked interactions with string theory [21]. The equivariant twisted K-theory of a Lie group G with respect to the adjoint action on itself was proven to be closely related to the Verlinde ring of positive energy representations of the loop group LG by Freed, Hopkins and Teleman [62].

From a homotopy theoretic viewpoint the twists of complex topological K-theory KU are classified by the generalized cohomology theory $gl_1(KU)$ associated to its unit spectrum. Its first group $[X, BGL_1(KU)]$ contains $[X, BBU(1)] \cong H^3(X, \mathbb{Z})$, which appears in the applications mentioned in the last paragraph. The existence of more general twistings was already pointed out in [8], but they were neglected since they had no obvious geometric interpretation in terms of bundles of operator algebras at that time. Nevertheless, in the setting of parametrized stable homotopy theory as developed by May and Sigurdsson [102] it is just as easy to deal with the whole space $BGL_1(KU)$ as it is to deal with $BBU(1)$. An equivalent approach to twisted generalized (co)homology theories was worked out by Ando, Blumberg, Gepner, Hopkins and Rezk in [5, 6]. It uses a generalized Thom spectrum associated to a map $X \to KU$-Line, where KU-Line is an ∞-groupoid with the homotopy type $BGL_1(KU)$.

As elegant as these approaches are, they give no hint towards an operator algebraic interpretation as in the case of ordinary twists. It is the goal of this paper to show that in fact the "higher twists" appear very naturally in the setting of operator algebras if one considers bundles of stabilized strongly self-absorbing C^*-algebras instead of bundles of compact operators. The class of these algebras was studied first by Toms and Winter [139, 150] and Dadarlat and Winter [51].

Their definition was motivated by the following observations: Certain algebras play a cornerstone role in Elliott's classification program of separable, nuclear, simple C^*-algebras: Tensorial absorption of the Cuntz algebra $\mathcal{O}_\infty$ detects purely infiniteness [119, Thm. 7.2.6]. Toms and Winter conjecture that in case of stably finite C^*-algebras, tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$ detects finite nuclear dimension [138]. Similarly, the Cuntz algebra $\mathcal{O}_2$ tensorially absorbs another algebra if and only if the latter is simple, separable, unital and nuclear [119, Thm. 7.1.2]. The definition of strongly self-absorbing isolates a property that all of these algebras share: They tensorially absorb themselves in a very strong sense (see Def. 5.2.1).

Surprisingly, this property has a lot of interesting topological and homotopy theoretic consequences: Given a strongly self-absorbing C^*-algebra D, the functor $X \mapsto K_*(C(X) \otimes D)$ is a multiplicative generalized cohomology theory, which can be represented by a commutative symmetric ring spectrum $KU^D_\bullet$ [49]. In particular, $KU^{\mathcal{O}_\infty}_\bullet$, $KU^{\mathcal{Z}}_\bullet$ both yield complex topological K-theory and if D is an infinite UHF-algebra, $KU^D_\bullet$ represents localizations of K-theory, such as $KU[1/p]$. It was proven in [47, 49] that $B\mathrm{Aut}(D \otimes \mathbb{K})$ is an infinite loop space, which comes with a map $B\mathrm{Aut}(D \otimes \mathbb{K}) \to BGL_1(KU^D)$. In case $D = \mathbb{C}$ we recover $B\mathrm{Aut}(\mathbb{K}) = BPU(H) \simeq BBU(1)$. In all other cases the map $\mathrm{Aut}(D \otimes \mathbb{K}) \to GL_1(KU^D)$ is an isomorphism on π_k for $k > 0$ and the inclusion $K_0(D)^\times_+ \to K_0(D)^\times$ on π_0. In particular, it is an equivalence if D is strongly self-absorbing and purely infinite [49, Thm. 4.6].

Based on these observations we will show in the current paper that the operator algebraic K-theory of section algebras of bundles with stabilized strongly self-absorbing fibers is indeed a valid model for higher twisted K-theory: In Section 2 we first construct a $KU^D_\bullet$-module spectrum $KU^{D,\mathrm{mod}}_\bullet$, on which the automorphism group $\mathrm{Aut}(D \otimes \mathbb{K})$ acts via maps of symmetric spectra. We then form a universal bundle of symmetric $KU^D_\bullet$-module spectra $\mathscr{K}U^D_\bullet$ over $B\mathrm{Aut}(D \otimes \mathbb{K})$. Sections of this bundle pulled back to a compact space X will provide the topological version of twisted K-theory. We prove that it has all the properties of a twisted generalized cohomology theory on pairs of compact topological spaces in Thm. 5.2.7 and compare it with its analytic counterpart, i.e. the operator algebraic K-theory of the corresponding section algebra.

In the next section we compare our definition with the twisted K-theory obtained from parametrized stable homotopy theory. We will not develop the analogue of the qf-model structure from [102] for symmetric spectra. Instead we consider an orthogonal ring spectrum $V \mapsto KU(V)$, which is equivalent to $KU^{\mathbb{C}}_{\bullet}$ when restricted to the symmetric group action. For any C^*-algebra A there is a KU-module spectrum KU^A, which is equivalent as a symmetric spectrum to $KU^{D,\mathrm{mod}}_{\bullet}$ for $A = D \otimes \mathbb{K}$. This allows us to compare our previous definition with the parametrized cohomology theories from [102, Ch. 20], which now merely amounts to citing the right theorems. In case $D = \{\mathbb{C}, \mathcal{O}_\infty, \mathcal{Z}\}$ no information is lost when switching to orthogonal spectra. However, there seems to be no obvious orthogonal counterpart for the symmetric ring spectrum $KU^D_{\bullet}$ in the case of general D, so we only retain the $KU^{\mathbb{C}}_{\bullet}$-module structure during this process.

Topological twisted K-homology is dual to twisted K-theory and can be defined as the homotopy groups of a generalized Thom spectrum $Mf_{\bullet}$ associated to a map $f\colon X \to BGL_1(KU)$. Depending on the setup, $Mf_{\bullet}$ can be constructed as an ∞-categorical colimit [5, 6], via the bar construction [102, Ch. 23] or as a smash product. We obtain $Mf_{\bullet}$ from the bundle of spectra $f^*\mathscr{K}U^D_{\bullet}$ by collapsing its zero section (Def. 5.3.1) in the spirit of [145]. Analytic K-homology on the other hand is defined via KK-theory. To compare the two functors on finite CW-complexes, we generalize the proof in [12] to the twisted case. In particular, we obtain an intermediate homology theory from framed bordism mapping to both of the theories. The map to analytic K-homology relies on a twisted version of Poincaré duality for bundles of C^*-algebras [57] and a twisted bordism invariant index map. This also yields a new proof of a similar result for ordinary twists, which was claimed in [145] for smooth manifolds and for finite CW-complexes in [146].

In the last part of this paper we give an explicit construction of higher twists over spaces of the form $X = \Sigma Y$, where we have $[\Sigma Y, BGL_1(KU^D)] \cong [Y, \mathrm{Aut}(D \otimes \mathbb{K})] \cong K^0(Y)^{\times}_{+}$. This yields continuous $C(X)$-algebras with fibers $D \otimes \mathbb{K}$ representing all possible higher twists in the suspension case, in particular for spheres. We then calculate the higher twisted K-groups of these spaces in terms of the K-theory of Y using a Mayer-Vietoris argument.

5.2 Twisted K-theory

5.2.1 Strongly self-absorbing C^*-algebras and $KU^D_{\bullet}$

The notion of strongly self-absorbing C^*-algebras was introduced by Toms and Winter in [139]. We will use the following definition, which is closer to topological applications and equivalent to the original one by [51, Thm. 2.2] and [150].

Definition 5.2.1. A unital C^*-algebra D is called *strongly self-absorbing* if it is separable and there exists a $*$-isomorphism $\psi\colon D \to D\otimes D$ and a path of unitaries $u\colon [0,1) \to U(D\otimes D)$ such that for all $d \in D$

$$\lim_{t\to 1} \|\psi(d) - u_t(d\otimes 1)u_t^*\| = 0 \ .$$

As mentioned in the introduction the main examples of strongly self-absorbing C^*-algebras are the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$, the Jiang-Su algebra $\mathcal{Z}$, infinite UHF algebras and tensor products of those. Their most important properties are summarized in [47, Thm. 2.1].

In the following D will always denote a strongly self-absorbing C^*-algebra. Let X be a compact Hausdorff space. The inverse isomorphism $\psi^{-1}\colon D\otimes D \to D$ together with the diagonal homomorphism $C(X)\otimes C(X) \cong C(X\times X) \to C(X)$ equips $K_*(C(X)\otimes D)$ with a ring structure. Since $\mathrm{Aut}(D)$ is contractible by [47, Thm. 2.3], the homotopy invariance of K-theory implies that this multiplication does not depend on the choice of ψ. Likewise, we obtain from the above property that the projection $[1_{C(X)\otimes D}] \in K_0(C(X)\otimes D)$ is the unit element of this graded ring.

The definition of the spectrum $KU^D_\bullet$ representing $X \mapsto K_*(C(X)\otimes D)$ will require graded C^*-algebras. We refer the reader to [18, VI.14] for an introduction. All tensor products that appear in the following are considered to be graded. Following [49] we denote the C^*-algebra $C_0(\mathbb{R})$ graded by odd and even functions by $\widehat{S}$. This is a coassociative, counital, coalgebra with comultiplication $\Delta\colon \widehat{S} \to \widehat{S}\otimes\widehat{S}$ and counit $\epsilon\colon \widehat{S} \to \mathbb{C}$ [53, 79]. It has the universal property that for any graded σ-unital C^*-algebra B there is a correspondence between essential graded $*$-homomorphisms $\varphi\colon \widehat{S} \to B$ and unbounded, self-adjoint, regular, odd operators $T\colon B \to B$ [141, Prop. 3.1]. Moreover, let $\mathbb{C}\ell_1$ be the complex Clifford algebra spanned by the even element 1 and the odd element c with $c^2 = 1$ and let $e \in \mathbb{K}$ be a fixed rank 1-projection.

Each strongly self-absorbing C^*-algebra D gives rise to a commutative symmetric ring spectrum $KU^D_\bullet$ [49]: Equip $D\otimes\mathbb{K}$ with the trivial grading and consider the sequence of topological spaces

$$KU^D_n = \hom_{\mathrm{gr}}(\widehat{S}, (\mathbb{C}\ell_1\otimes D\otimes\mathbb{K})^{\otimes n}) \ ,$$

where each of the homomorphism spaces is equipped with the pointwise norm topology and pointed by the zero homomorphism. Let $\mu_{m,n}$ be the following multiplication map

$$\mu_{m,n}\colon KU^D_m \wedge KU^D_n \to KU^D_{m+n} \quad ; \quad \varphi\wedge\psi \mapsto (\varphi\otimes\psi)\circ\Delta$$

By the universal property of $\widehat{S}$ the unbounded, self-adjoint, regular, odd multiplier $t \mapsto t\,c$ on $C_0(\mathbb{R},\mathbb{C}\ell_1)$ defines a $*$-homomorphism $\widehat{S} \to C_0(\mathbb{R},\mathbb{C}\ell_1)$. Tensor product with $\mathbb{C} \to D\otimes\mathbb{K}$

given by $1 \mapsto 1 \otimes e$ induces a map $\widehat{\eta}_1 \colon \widehat{S} \to C_0(\mathbb{R}, \mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})$, which in turn induces $\eta_1 \colon S^1 \to KU_1^D$. Forming products we obtain $\eta_n \colon S^n \to KU_n^D$. It was shown in [49, Thm. 4.2] that $(KU_\bullet^D, \mu_{\bullet,\bullet}, \eta_\bullet)$ gives a commutative symmetric ring spectrum, which represents the functor $X \mapsto K_\bullet(C(X) \otimes D)$.

We will also need the definition of a module spectrum over $KU_\bullet^D$.

Definition 5.2.2. Let $(R_\bullet, \mu_{\bullet,\bullet}, \eta_\bullet)$ be a commutative symmetric ring spectrum. A *module spectrum* is a sequence of pointed spaces $(M_n)_{n\geq 0}$ with basepoint preserving continuous left action of the symmetric group Σ_n on M_n for each $n \geq 0$ together with $\Sigma_m \times \Sigma_n$-equivariant action maps $\alpha_{m,n} \colon M_m \wedge R_n \to M_{m+n}$ for all $n, m \geq 0$, such that the following diagrams commute:

$$\begin{tikzcd}
M_\ell \wedge R_m \wedge R_n \arrow[r, "\alpha_{\ell,m} \wedge \mathrm{id}_{R_n}"] \arrow[d, "\mathrm{id}_{M_\ell} \wedge \mu_{m,n}"'] & M_{\ell+m} \wedge R_n \arrow[d, "\alpha_{\ell+m,n}"] \\
M_\ell \wedge R_{m+n} \arrow[r, "\alpha_{\ell,m+n}"'] & M_{\ell+m+n}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_n \wedge S^0 \arrow[r, "\mathrm{id}_{M_n} \wedge \eta_0"] \arrow[dr] & M_n \wedge R_0 \arrow[d, "\alpha_{n,0}"] \\
 & M_n
\end{tikzcd}$$

where the map $M_n \wedge S^0 \to M_n$ in the right diagram is the canonical one.

We now define the module spectrum $KU_\bullet^{D,\mathrm{mod}}$ over $KU_\bullet^D$, which carries an $\mathrm{Aut}(D \otimes \mathbb{K})$-action that is compatible with the action maps.

Definition 5.2.3. Let D be a strongly self-absorbing C^*-algebra. Let $KU_\bullet^{D,\mathrm{mod}}$ be the following sequence of spaces

$$KU_n^{D,\mathrm{mod}} = \hom_{\mathrm{gr}}(\widehat{S}, D \otimes \mathbb{K} \otimes (\mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})^{\otimes n}) \ ,$$

where as above the graded homomorphisms are equipped with the point-norm topology and $D \otimes \mathbb{K}$ is considered to be trivially graded. Let

$$\alpha_{m,n} \colon KU_m^{D,\mathrm{mod}} \wedge KU_n^D \to KU_{m+n}^{D,\mathrm{mod}} \quad ; \quad \varphi \wedge \psi \mapsto (\varphi \otimes \psi) \circ \Delta$$

Observe that the space $KU_n^{D,\mathrm{mod}}$ carries a natural left action of the group $\mathrm{Aut}(D \otimes \mathbb{K})$ in the following way: Let $\beta \in \mathrm{Aut}(D \otimes \mathbb{K})$ and $\varphi \in KU_n^{D,\mathrm{mod}}$, then we set $\beta \cdot \varphi = (\beta \otimes \mathrm{id}_{(\mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})^{\otimes n}}) \circ \varphi$. Moreover there is a natural Σ_n-action on $KU_n^{D,\mathrm{mod}}$ by permuting the n factors of the graded tensor product $(\mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})^{\otimes n}$ using the Koszul sign rule just as in $KU_\bullet^D$.

Theorem 5.2.4. *The pair $(KU_\bullet^{D,\mathrm{mod}}, \alpha_{\bullet,\bullet})$ forms a module spectrum over the ring spectrum $KU_\bullet^D$, which is equivalent to $KU_\bullet^D$ as a module spectrum. The module structure is compatible* *with the action of* $\mathrm{Aut}(D \otimes \mathbb{K})$ *described above. All structure maps* $KU_n^{D,\mathrm{mod}} \to \Omega KU_{n+1}^{D,\mathrm{mod}}$ *are weak homotopy equivalences for* $n \geq 1$ *and the coefficients are given by*

$$\pi_i(KU_\bullet^{D,\mathrm{mod}}) = K_i(D) \ .$$

Proof. It is clear that the Σ_n-action preserves the basepoint of $KU_n^{D,\mathrm{mod}}$. It is a consequence of the coassociativity of $\Delta \colon \widehat{S} \to \widehat{S} \otimes \widehat{S}$ that the associativity diagram in Def. 5.2.3 commutes. Since η_0 sends the non-basepoint of S^0 to the counit $\epsilon \colon \widehat{S} \to \mathbb{C}$, the unitality diagram also commutes. The group Σ_n acts on $KU_n^{D,\mathrm{mod}}$ from the left. This implies the equivariance of $\alpha_{m,n}$ with respect to the $\Sigma_m \times \Sigma_n$-action on both sides. This proves that $(KU_\bullet^{D,\mathrm{mod}}, \alpha_{\bullet,\bullet})$ is in fact a symmetric $KU_\bullet^D$-module spectrum.

Let $\beta \in \mathrm{Aut}(D \otimes \mathbb{K})$, $\varphi \in KU_m^{D,\mathrm{mod}}$, $\psi \in KU_n^D$, then we have

$$\alpha_{m,n}((\beta \cdot \varphi) \wedge \psi) = (\beta \otimes \mathrm{id}) \circ (\varphi \otimes \psi) \circ \Delta = \beta \cdot \alpha_{m,n}(\varphi \wedge \psi) \ ,$$

which is the claimed compatibility with the action. To see that $KU_\bullet^{D,\mathrm{mod}}$ is equivalent to $KU_\bullet^D$ as a module spectrum, consider the Σ_n-equivariant map $\theta_n \colon KU_n^D \to KU_n^{D,\mathrm{mod}}$ given by $\theta_n(\varphi)(f) = 1 \otimes e \otimes \varphi(f)$ for $f \in \widehat{S}$, $\varphi \in KU_n^D$. Associativity of the graded tensor product implies $\alpha_{m,n}(\theta_m(\varphi) \wedge \psi) = \theta_{m+n}(\mu_{m,n}(\varphi \wedge \psi))$ for $\varphi \in KU_m^D$, $\psi \in KU_n^D$, i.e. θ_n is a map of $KU_\bullet^D$-module spectra. Since D is strongly self-absorbing, the homomorphism $D \otimes \mathbb{K} \to (D \otimes \mathbb{K})^{\otimes 2}$ given by $a \mapsto 1 \otimes e \otimes a$ is homotopic to an isomorphism. Therefore θ_n is a weak equivalence for $n \geq 1$. The statement about the structure maps and the coefficients follows from [49, Thm. 4.2]. This implies that $KU_\bullet^{D,\mathrm{mod}}$ is semistable, therefore $\theta_\bullet$ is actually an equivalence of symmetric spectra. □

5.2.2 Bundles of $KU_\bullet^D$-module spectra

The advantage of $KU_\bullet^{D,\mathrm{mod}}$ over $KU_\bullet^D$ is that it carries a natural left action of the group $\mathrm{Aut}(D \otimes \mathbb{K})$, which enables us to form a bundle of symmetric spectra over $B\mathrm{Aut}(D \otimes \mathbb{K})$.

Definition 5.2.5. Let $E\mathrm{Aut}(D \otimes \mathbb{K}) \to B\mathrm{Aut}(D \otimes \mathbb{K})$ be the universal principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle and denote the left action of $\mathrm{Aut}(D \otimes \mathbb{K})$ on $KU_n^{D,\mathrm{mod}}$ by λ_n. We define

$$\mathscr{K}U_n^D = E\mathrm{Aut}(D \otimes \mathbb{K}) \times_{\lambda_n} KU_n^{D,\mathrm{mod}} \ ,$$

let $\pi_n \colon \mathscr{K}U_n^D \to B\mathrm{Aut}(D \otimes \mathbb{K})$ be the projection map and let $\sigma_n \colon B\mathrm{Aut}(D \otimes \mathbb{K}) \to \mathscr{K}U_n^D$ be the zero section. Note that $\mathrm{Aut}(D \otimes \mathbb{K})$ also acts on the loop space $\Omega KU_\bullet^{D,\mathrm{mod}}$ preserving its basepoint. We denote the corresponding bundle by $\Omega \mathscr{K}U_\bullet^D$ and we have structure maps $\kappa_n \colon \mathscr{K}U_n^D \to \Omega \mathscr{K}U_{n+1}^D$ induced by $KU_n^{D,\mathrm{mod}} \to \Omega KU_{n+1}^{D,\mathrm{mod}}$.

Let $\mathcal{T}op_D^{2,\mathrm{cpt}}$ be the category of triples (X, B, f), where X is a compact Hausdorff space, $B \subset X$ is a closed subspace and $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ is a continuous map classifying a principal bundle $\mathcal{P}_f = f^*E\mathrm{Aut}(D \otimes \mathbb{K})$ over X with an associated bundle of C^*-algebras $\mathcal{A} = \mathcal{A}_f \to X$ with fiber $D \otimes \mathbb{K}$. A morphism in $\mathcal{T}op_D^{2,\mathrm{cpt}}$ is given by a pair $(\varphi, \hat{\varphi})$, where φ is a continuous map of pairs $(X, B) \to (X', B')$ and $\hat{\varphi}$ is a map of principal bundles $\mathcal{P}_f \to \mathcal{P}_{f'}$ covering φ. Define $c(X, B) = X \amalg B \times [0, 1)/\sim$, where the equivalence relation identifies $(b, 0) \in B \times [0, 1)$ with $b \in X$. The bundle $\mathcal{P}_f$ extends in a canonical way to a bundle $c\mathcal{P}_f \to c(X, B)$ with an associated bundle of C^*-algebras $c\mathcal{A} \to c(X, B)$. We have a short exact sequence of section algebras

$$0 \to SC(B, \mathcal{A}|_B) \to C_0(c(X, B), c\mathcal{A}) \to C(X, \mathcal{A}) \to 0 \tag{5.1}$$

where $SC(B, \mathcal{A}|_B) = C_0((0, 1)) \otimes C(B, \mathcal{A}|_B)$ is the suspension. We define the ordinary K-theory of the pair (X, B) with coefficients in D by $K_D^i(X, B) = K_i(C_0(c(X, B)) \otimes D)$. This is justified by the fact that for $D = \mathbb{C}$ this group is the K-theory of the mapping cone of the inclusion $B \subseteq X$ relative to its tip. Since D is strongly self-absorbing $K_D^*(X, B)$ is a graded ring (which is unital if $B = \emptyset$). Twisted K-theory is a functor from $\mathcal{T}op_D^{2,\mathrm{cpt}}$ to graded abelian groups in such a way that it maps a triple (X, B, f) to a module over the ordinary K-theory of (X, B) with coefficients in D.

Definition 5.2.6. Let $(X, B, f) \in \mathrm{Obj}(\mathcal{T}op_D^{2,\mathrm{cpt}})$, let $\mathcal{P} = f^*E\mathrm{Aut}(D \otimes \mathbb{K}) \to X$ and let $\mathrm{Sec}_n(X, B; \mathcal{P})$ be the space of all pairs (τ, H), where $\tau\colon X \to f^*\mathscr{K}U_n^D$ is a section and $H\colon B \times I \to (f|_B \times \mathrm{id}_I)^*\mathscr{K}U_n^D$ is another section which satisfies $H_0 = \tau|_B$ and $H_1(a) = \sigma_n(f(a))$ (where σ_n denotes the zero section from Def. 5.2.5). This space is pointed by $(\sigma_n \circ f, \mathrm{const}_\sigma)$, where const_σ is the constant homotopy on $\sigma_n \circ f|_B$. Observe that the structure maps of $\mathscr{K}U_n^D$ induce continuous maps

$$\mathrm{Sec}_n(X, B; \mathcal{P}) \to \Omega\mathrm{Sec}_{n+1}(X, B; \mathcal{P})$$

Define the *twisted K-group* $K_{\mathcal{P}}^i(X, B)$ of (X, B, f) by

$$K_{\mathcal{P}}^i(X, B) = \mathrm{colim}_n\, \pi_{i+n}(\mathrm{Sec}_n(X, B; \mathcal{P}))\ .$$

Let $K_{\mathcal{P}}^i(X) = K_{\mathcal{P}}^i(X, \emptyset)$. Note that $\mathrm{Sec}_n(X, \emptyset; \mathcal{P})$ is just the space of sections of $f^*\mathscr{K}U_n^D$ pointed by the zero section.

Theorem 5.2.7. *Let $(X, B, f) \in \mathrm{Obj}(\mathcal{T}op_D^{2,\mathrm{cpt}})$ and let $\mathcal{P} = f^*E\mathrm{Aut}(D \otimes \mathbb{K})$. Twisted K-theory has the following properties:*

(a) The structure maps κ_n of the bundle $\mathscr{K}U_n^D$ are weak equivalences for $n \geq 1$. Therefore $K_{\mathcal{P}}^m(X, B) \cong \pi_{m+1}(\mathrm{Sec}_1(X, B; \mathcal{P}))$.

(b) $K^\bullet \colon \mathcal{T}op_D^{2,\mathrm{cpt}} \to \mathrm{GrAb}$ *is a homotopy invariant contravariant functor to graded abelian groups, such that* $K^\bullet_{\mathcal{P}}(X,B)$ *is a module over* $K^\bullet_D(X,B)$*.*

(c) *Topological and analytic twisted K-theory are naturally isomorphic, i.e. we have*

$$K^m_{\mathcal{P}}(X,B) \cong K_m(C_0(c(X,B), c\,\mathcal{A}))$$

with $\mathcal{A}$ and $c\,\mathcal{A}$ as above. In particular, there is a natural isomorphism $K^m_{\mathcal{P}}(X) \cong K_m(C(X,\mathcal{A}))$ *and a trivialization of $\mathcal{P}$ induces* $K^m_{\mathcal{P}}(X,B) \cong K^m_D(X,B)$*.*

(d) *Let* $i \colon (X,\emptyset) \to (X,B)$ *and* $j \colon B \to X$ *be given by inclusion. There is a six-term exact sequence of the form*

$$\begin{tikzcd}
K^0_{\mathcal{P}}(X,B) \arrow[r, "i^*"] & K^0_{\mathcal{P}}(X) \arrow[r, "j^*"] & K^0_{\mathcal{P}|_B}(B) \arrow[d, "\partial_{01}"] \\
K^1_{\mathcal{P}|_B}(B) \arrow[u, "\partial_{10}"] & K^1_{\mathcal{P}}(X) \arrow[l, "j^*"'] & K^1_{\mathcal{P}}(X,B) \arrow[l, "i^*"']
\end{tikzcd}$$

(e) $K^m_{\mathcal{P}}$ *satifies excision in the sense that for every open set $U \subset B$ such that the closure of U is in the interior of B we have* $K^m_{\mathcal{P}}(X,B) \cong K^m_{\mathcal{P}}(X \setminus U, B \setminus U)$ *.*

(f) *Let $X = V_0 \cup V_1$ for closed subsets V_i, such that their interiors still cover X. Let* $i_k \colon V_k \to X$ *and let* $j_k \colon V_0 \cap V_1 \to V_k$ *be given by inclusion. Then there is a six-term Mayer-Vietoris sequence:*

$$\begin{tikzcd}
K^0_{\mathcal{P}}(X) \arrow[r, "(i_0^*, i_1^*)"] & K^0_{\mathcal{P}|_{V_0}}(V_0) \oplus K^0_{\mathcal{P}|_{V_1}}(V_1) \arrow[r, "j_0^* - j_1^*"] & K^0_{\mathcal{P}|_{V_0 \cap V_1}}(V_0 \cap V_1) \arrow[d] \\
K^1_{\mathcal{P}|_{V_0 \cap V_1}}(V_0 \cap V_1) \arrow[u] & K^1_{\mathcal{P}|_{V_0}}(V_0) \oplus K^1_{\mathcal{P}|_{V_1}}(V_1) \arrow[l, "j_0^* - j_1^*"'] & K^1_{\mathcal{P}}(X) \arrow[l, "(i_0^*, i_1^*)"']
\end{tikzcd}$$

Proof. By the same argument as in the proof of [49, Thm. 4.2] the maps $KU_n^{D,\mathrm{mod}} \to \Omega KU_{n+1}^{D,\mathrm{mod}}$ are weak equivalences for $n \geq 1$. The first statement of (a) follows from the long exact sequence for the fibration $KU_n^{D,\mathrm{mod}} \to \mathscr{K}U_n^D \to B\mathrm{Aut}(D \otimes \mathbb{K})$.

Let $(\tau, H) \in \mathrm{Sec}_n(X,B;\mathcal{P})$, let $C_n = (\mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})^{\otimes n}$. The section τ yields an element in $\hom_{\mathrm{gr}}(\widehat{S}, C(X,\mathcal{A}) \otimes C_n)$. Similarly, H yields a homomorphism in $\hom_{\mathrm{gr}}(\widehat{S}, C_0(B \times [0,1), \pi_B^*\mathcal{A}) \otimes C_n)$. The algebra $C_0(c(X,B), c\,\mathcal{A})$ is the pullback of $C(X,\mathcal{A})$ and $C_0(B \times [0,1), \pi_B^*\mathcal{A})$ along the restriction maps to $C(B, \mathcal{A}|_B)$. Thus, the definition of $\mathrm{Sec}_n(X,B;\mathcal{P})$

ensures that both homomorphisms piece together to form a point in

$$K_n = \hom_{\mathrm{gr}}(\widehat{S}, C_0(c(X,B), c\,\mathcal{A}) \otimes (\mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})^{\otimes n}) \ .$$

This construction provides a homeomorphism between $\mathrm{Sec}_n(X,B;\mathcal{P})$ and K_n with respect to which $\mathrm{Sec}_n(X,B;\mathcal{P}) \to \Omega\,\mathrm{Sec}_{n+1}(X,B;\mathcal{P})$ translates to a map $K_n \to \Omega K_{n+1}$. The latter is essentially given by Bott periodicity and tensor product with the projection $1 \otimes e \in D \otimes \mathbb{K}$ as in the proof of [49, Thm. 4.2]. Thus, $K_n \to \Omega K_{n+1}$ is a weak equivalence for $n \geq 1$, which proves the isomorphism in (a). By [141, Thm. 4.7] and Bott periodicity, we have natural isomorphisms

$$\begin{aligned}\pi_{m+1}(K_1) &= \pi_0(\hom_{\mathrm{gr}}(\widehat{S}, S^m C_0(c(X,B), c\,\mathcal{A}) \otimes C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes D \otimes \mathbb{K})) \\ &\cong K_0(S^m C_0(c(X,B), c\,\mathcal{A})) \cong K_m(C_0(c(X,B), c\,\mathcal{A}))\end{aligned}$$

If $B = \emptyset$ we can identify $c(X,B) = X$ and $c\,\mathcal{A} = \mathcal{A}$. This proves (c). Since K_m is homotopy invariant and $K_\bullet(C_0(c(X,B), c\,\mathcal{A}))$ is a module over $K_\bullet(C_0(c(X,B)) \otimes D)$, it also proves (b). The algebra $C_0(c(X,B), c\,\mathcal{A})$ fits into a short exact sequence

$$0 \to SC(B, \mathcal{A}|_B) \to C_0(c(X,B), c\,\mathcal{A}) \to C(X, \mathcal{A}) \to 0 \ . \tag{5.2}$$

Using the isomorphism from (c), the six-term exact sequence associated to the above short exact sequence is of the form given in (d), but we have to check that the map $K_0(C(X,\mathcal{A})) \to K_1(SC(B, \mathcal{A}|_B)) \cong K_0(C(B, \mathcal{A}|_B))$ (with the indicated isomorphism given by Bott periodicity) coincides with the one induced by the inclusion $B \to X$. We can restrict (5.2) to the sequence

$$0 \to SC(B, \mathcal{A}|_B) \to C_0(B \times [0,1), c\,\mathcal{A}|_{B \times [0,1)}) \to C(B, \mathcal{A}|_B) \to 0 \ . \tag{5.3}$$

in which the middle algebra is contractible. Naturality of the boundary map yields the commutative diagram

$$\begin{tikzcd} K_0(C(X,\mathcal{A})) \arrow[r] \arrow[d] & K_1(SC(B, \mathcal{A}|_B)) \arrow[d, equal] \\ K_0(C(B, \mathcal{A}|_B)) \arrow[r, "\cong"'] & K_1(SC(B, \mathcal{A}|_B)) \end{tikzcd}$$

where the vertical arrow is induced by the inclusion and the isomorphism from the six-term sequence associated to (5.3) coincides with the Bott isomorphism. This finishes the proof of (d). The Mayer-Vietoris sequence (f) is a consequence of the isomorphism in (c) and the fact that $C(X, \mathcal{A})$ fits into the pullback diagram

$$\begin{tikzcd} C(X, \mathcal{A}) \arrow[r] \arrow[d] & C(V_0, \mathcal{A}|_{V_0}) \arrow[d] \\ C(V_1, \mathcal{A}|_{V_1}) \arrow[r] & C(V_0 \cap V_1, \mathcal{A}|_{V_0 \cap V_1}) \end{tikzcd}$$

by [18, Thm. 21.2.3]. That excision, i.e. (e), holds for the twisted K-functor follows from the contractibility of the algebra $C_0(U \times [0,1), c\,\mathcal{A}|_{U\times[0,1)})$ and the long exact sequence associated to

$$0 \to C_0(U \times [0,1), c\,\mathcal{A}|_{U\times[0,1)}) \to C_0(c(X,B), c\,\mathcal{A}) \to C_0(c(X\backslash U, B\backslash U), c\,\mathcal{A}|_{c(X\backslash U, B\backslash U)}) \to 0$$

□

Remark 5.2.8. It is straightforward to check that the $K^\bullet_D(X,B)$-module structure agrees with the one obtained from the $KU^D_\bullet$-module structure of the spectrum $KU^{D,\mathrm{mod}}_\bullet$.

Remark 5.2.9. There is an alternative way to prove the six-term exact sequence from Thm. 5.2.7 (d), which is closer to stable homotopy theory: Let $K_n^{(X,B)} = K_n$ be the space from the proof of Thm. 5.2.7 and let $K_n^X := K_n^{(X,\emptyset)}$. One can show that the homotopy fiber of the natural restriction map $K_n^{(X,B)} \to K_n^X$ is weakly homotopy equivalent to ΩK_n^B. Moreover, the homotopy fiber of $K_n^X \to K_n^B$ is $K_n^{(X,B)}$. Thus, we can look at the fibration sequence

$$\begin{tikzcd} & & \Omega K_n^B \arrow[r] & K_n^{(X,B)} \arrow[r] & K_n^X \arrow[r] & K_n^B \\ K_n^{(X,B)} \arrow[r] & K_{n-1}^X \arrow[r] & K_{n-1}^B \arrow[u] & & & \end{tikzcd}$$

in which the vertical map is a weak equivalence for $n > 1$. The exact sequence in (d) now follows from Bott periodicity and the long exact sequence of homotopy groups.

5.2.3 Parametrized stable homotopy theory

A slightly different approach to twisted K-theory is based on parametrized stable homotopy theory [102]. To compare it with the definition given above, we will develop an orthogonal version of the bundle of symmetric $KU_\bullet$-module spectra. It has the advantage that it yields parametrized spectra over $B\mathrm{Aut}(A)$ for arbitrary C^*-algebras A. The price one has to pay for this is that its fibers are only $KU_\bullet = KU^{\mathbb{C}}_\bullet$-modules in a natural way. Nevertheless, in case $A = \mathcal{O}_\infty \otimes \mathbb{K}$, the space $B\mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})$ has the homotopy type of $BGL_1(KU)$ [49, Thm. 4.6] and the resulting theory is an extension of the above symmetric version representing twisted K-theory including all higher twists.

Definition 5.2.10. Let A be a σ-unital C^*-algebra. Given a finite dimensional inner product space V, we define

$$KU^A(V) = \hom_{\mathrm{gr}}(\widehat{S}, A \otimes \mathbb{C}\ell(V) \otimes \mathbb{K}(L^2(V))) \ ,$$

which is pointed by the zero homomorphism.

We denote $KU^{\mathbb{C}}(V)$ by $KU(V)$. The isometric isomorphism $L^2(V \oplus W) \cong L^2(V) \otimes L^2(W)$ induces an isomorphism of C^*-algebras $\mathbb{K}(L^2(V)) \otimes \mathbb{K}(L^2(W)) \to \mathbb{K}(L^2(V \oplus W))$. Moreover, we have an isomorphism $\mathbb{C}\ell(V) \otimes \mathbb{C}\ell(W) \cong \mathbb{C}\ell(V \oplus W)$ of graded C^*-algebras. Just as in Def. 5.2.3 we therefore obtain a multiplication map

$$\mu_{V,W} \colon KU(V) \wedge KU(W) \to KU(V \oplus W) \ .$$

Let S^V be the one-point compactification of V. The canonical linear map $V \to \mathbb{C}\ell(V)$ defines an unbounded, self-adjoint, regular, odd multiplier of $C_0(V, \mathbb{C}\ell(V))$, which induces the map $\eta_V \colon S^V \to \hom_{\mathrm{gr}}(\widehat{S}, \mathbb{C}\ell(V) \otimes \mathbb{K}(L^2(V)))$. It was shown in [80] that $(KU(\bullet), \mu_{\bullet,\bullet}, \eta_\bullet)$ is an orthogonal ring spectrum.

As in the symmetric case there is an action $\alpha_{V,W} \colon KU^A(V) \wedge KU(W) \to KU^A(V \oplus W)$ of KU on KU^A given by $\varphi \wedge \psi \mapsto (\varphi \otimes \psi) \circ \Delta$, where we suppress the above isomorphisms in the notation. Apart from the action of KU, $KU^A(V)$ also carries a (left) action of the group $\mathrm{Aut}(A)$. We will omit the proof of the following lemma, which is completely analogous to that of Thm. 5.2.4.

Lemma 5.2.11. *The pair $(KU^A(\bullet), \alpha_{\bullet,\bullet})$ forms an orthogonal module spectrum over KU. The module structure is compatible with the action of* $\mathrm{Aut}(A)$ *described above. The homotopy groups of $KU^A(\bullet)$ are given by $\pi_i(KU^A(\bullet)) = K_i(A)$.*

We will consider the category of spaces X over a fixed base space B, such that each fiber is continuously pointed. These are called ex-spaces.

Definition 5.2.12. An *ex-space* over a topological base space B is a topological space X together with continuous maps $\pi \colon X \to B$ and $\sigma \colon B \to X$ such that $\pi \circ \sigma = \mathrm{id}_B$.

There is a good notion of smash product $X \wedge_B Y$ for ex-spaces defined by forming smash products fiberwise, and there is also an internal Hom-functor $F_B(X, Y)$ adjoint to $\wedge_B$. It associates to two ex-spaces X and Y over B another ex-space $F_B(X, Y)$, such that the fiber over $b \in B$ agrees with continuous based maps $X_b \to Y_b$ [102, Def. 1.3.12]. Let $f \colon A \to B$ be a continuous map. As explained in [102, Sec. 2.1] there is a pullback functor f^*, which has a right adjoint f_* and a left adjoint $f_!$. Let $r \colon B \to *$ be the canonical map. The parametrized loop space functor is given by $\Omega_B X = F_B(r^* S^1, X)$. Any space X over B can be turned into an ex-space by adding a disjoint basepoint in each fiber, i.e. by setting $X_+ = B \amalg X$ with the obvious projection and section. Given a real finite dimensional inner product space V let

$$\mathscr{K}U^A(V) = E\mathrm{Aut}(A) \times_\lambda KU^A(V) \ .$$

Together with the projection $\pi\colon \mathscr{K}U^A(V) \to B\mathrm{Aut}(A)$ and zero section $\sigma\colon B\mathrm{Aut}(A) \to \mathscr{K}U^A(V)$, this is an ex-space over $B\mathrm{Aut}(A)$. Since $\alpha_{V,W}$ is equivariant with respect to the action of $\mathrm{Aut}(A)$, it extends to a map

$$\bar{\alpha}_{V,W}\colon \mathscr{K}U^A(V) \wedge_{B\mathrm{Aut}(A)} KU(W) \to \mathscr{K}U^A(V \oplus W)$$

The following corollary is the outcome of [102, Sec. 20.2].

Corollary 5.2.13. *The family $(\mathscr{K}U^A(\bullet), \bar{\alpha}_{\bullet,\bullet})$ is a parametrized KU-module spectrum over $B\mathrm{Aut}(A)$ in the sense of [102, Def. 14.1.1]. Given an ex-space (X, f, σ) over $B = B\mathrm{Aut}(A)$, let $\mathcal{P} = f^*E\mathrm{Aut}(A)$ and*

$$\widetilde{K}^{i,\mathrm{para}}_{\mathcal{P}}(X) = \pi_i(r_*F_B(X, \mathscr{K}U^A(\bullet))) \ .$$

This is a (reduced) parametrized cohomology theory in the sense of [102, Def. 20.1.2].

Theorem 5.2.14. *Let D be a strongly self-absorbing C^*-algebra, let $A = D \otimes \mathbb{K}$.*

(a) If $D \in \{\mathcal{O}_\infty, \mathcal{Z}, \mathbb{C}\}$, then the symmetric ring spectrum obtained from $n \mapsto KU(\mathbb{R}^n)$ by restricting the $O(n)$-action to a Σ_n-action is equivalent to $KU^D_\bullet$ as a ring spectrum.

(b) The symmetric spectrum obtained from $KU^A(\bullet)$ by restricting the $O(n)$-action to a Σ_n-action is equivalent to $KU^{D,\mathrm{mod}}_\bullet$ as a $KU^{\mathbb{C}}_\bullet$-module spectrum and the equivalence commutes with the group action of $\mathrm{Aut}(D \otimes \mathbb{K})$ *on both sides.*

(c) Let $(X, B, f) \in \mathcal{T}op^{2,\mathrm{cpt}}_D$, let $\mathcal{P} = f^*E\mathrm{Aut}(A)$, let $\iota_+\colon B_+ \to X_+$ be the induced inclusion of ex-spaces and denote by $c(\iota_+)$ the (parametrized) mapping cone of ι_+. The parametrized twisted K-groups $K^{i,\mathrm{para}}_{\mathcal{P}}(X, B) = \widetilde{K}^{i,\mathrm{para}}_{\mathcal{P}}(c(\iota_+))$ are naturally isomorphic to the twisted K-groups from Def. 5.2.6, i.e. $K^{i,\mathrm{para}}_{\mathcal{P}}(X, B) \cong K^i_{\mathcal{P}}(X, B)$.

Proof. Observe that $\mathbb{C}\ell(\mathbb{R}^n) \otimes \mathbb{K}(L^2(\mathbb{R}^n)) \cong (\mathbb{C}\ell_1 \otimes \mathbb{K}(L^2(\mathbb{R})))^{\otimes n} = (\mathbb{C}\ell_1 \otimes \mathbb{K})^{\otimes n}$, where the isomorphism can be chosen to be compatible with the action of Σ_n on both sides. Let $\kappa\colon \mathbb{C}\ell_1 \otimes \mathbb{K} \to \mathbb{C}\ell_1 \otimes D \otimes \mathbb{K}$ be the map that sends $v \otimes T$ to $v \otimes 1_D \otimes T$. This induces a natural map $KU(\mathbb{R}^n) \to KU^D_n$, which sends $\varphi\colon \widehat{S} \to (\mathbb{C}\ell_1 \otimes \mathbb{K})^{\otimes n}$ to the homomorphism $\hat{\varphi}\colon \widehat{S} \to (\mathbb{C}\ell_1 \otimes D \otimes \mathbb{K})^{\otimes n}$ with $\hat{\varphi} = \kappa^{\otimes n} \circ \varphi$. On homotopy groups it induces the homomorphism $K_*(\mathbb{C}) \to K_*(D)$ given by $[p] \mapsto [p \otimes 1_D]$. This is an isomorphism for $D \in \{\mathcal{O}_\infty, \mathcal{Z}, \mathbb{C}\}$. Since both sides are semistable spectra, they are equivalent. We have

$$(\hat{\varphi} \otimes \hat{\psi}) \circ \Delta = (\kappa^{\otimes n} \otimes \kappa^{\otimes m}) \circ (\varphi \otimes \psi) \circ \Delta = \kappa^{\otimes n+m} \circ (\varphi \otimes \psi) \circ \Delta \ .$$

Moreover, composition of the map $\eta_n^{KU} \colon S^n \to KU(\mathbb{R}^n)$ with the above map $KU(\mathbb{R}^n) \to KU_n^D$ yields $\eta_n^{KU^D}$, i.e. $\varphi \mapsto \hat{\varphi}$ preserves units. This proves (a).

The proof of (b) is similar: Again we identify $\mathbb{C}\ell(\mathbb{R}^n) \otimes K(L^2(\mathbb{R}^n))$ with $(\mathbb{C}\ell_1 \otimes \mathbb{K})^{\otimes n}$. The map $KU^A(\mathbb{R}^n) \to KU_n^{D,\mathrm{mod}}$ sends φ to $\tilde{\varphi} = (\mathrm{id}_{D\otimes\mathbb{K}} \otimes \kappa^{\otimes n}) \circ \varphi$, which induces an isomorphism on homotopy groups and therefore provides a stable equivalence. The map is compatible with the left action of $\mathrm{Aut}(D \otimes \mathbb{K})$. The following diagram commutes

$$\begin{tikzcd}
KU^A(\mathbb{R}^n) \wedge KU(\mathbb{R}^m) \arrow[r] \arrow[d, "\alpha_{\mathbb{R}^n,\mathbb{R}^m}"'] & KU_n^{D,\mathrm{mod}} \wedge KU_m^{\mathbb{C}} \arrow[d, "\alpha^{\mathbb{C}}_{n,m}"] \\
KU^A(\mathbb{R}^{n+m}) \arrow[r] & KU_{n+m}^{D,\mathrm{mod}}
\end{tikzcd}$$

where $\alpha^{\mathbb{C}}_{n,m}$ is induced by the map of spectra $KU^{\mathbb{C}}_{\bullet} \to KU^{D}_{\bullet}$. This proves that the stable equivalence intertwines the KU-action on the left hand side with the one by $KU^{\mathbb{C}}_{\bullet}$ on the right.

The mapping cone $c(\iota_+)$ is given by $B_+ \wedge_{B\mathrm{Aut}(A)} I \cup_{\iota_+} X_+$, where I has basepoint 0. This space is homeomorphic as an ex-space to a quotient $(M(\iota)_+)/{\sim}$ of the mapping cylinder, where the equivalence relation identifies all points $(b,0) \in M(\iota)$ in a fiber of $f|_B$ with the disjoint basepoint in that fiber. Let $g \colon Y \to C$ be a space over $C = B\mathrm{Aut}(A)$. By local triviality of the bundle $\mathscr{K}U^A(\mathbb{R}^n)$ and [102, Prop. 2.2.2, eqn. (2.2.7)] the mapping space $F_C(Y_+, \mathscr{K}U^A(\mathbb{R}^n))$ is homeomorphic to $\mathrm{Sec}(Y, g^*\mathscr{K}U^A_{\mathbb{R}^n})$. It follows from this that $F_C(c(\iota_+), \mathscr{K}U^A(\mathbb{R}^n))$ is homeomorphic to the subspace of $\mathrm{Sec}(M(\iota), f^*\mathscr{K}U^A(\mathbb{R}^n))$ consisting of sections $\tau \colon M(\iota) \to f^*\mathscr{K}U^A(\mathbb{R}^n)$ with $\tau(b,0) = \sigma_n(f(b))$, where $\sigma_n \colon B\mathrm{Aut}(A) \to \mathscr{K}U^A(\mathbb{R}^n)$ is the zero section. For $(X,B) \in \mathcal{T}op_D^{2,\mathrm{cpt}}$ this space is homeomorphic to

$$\widehat{K}_n = \hom_{\mathrm{gr}}(\widehat{S}, C_0(c(X,B), c\,\mathcal{A}) \otimes (\mathbb{C}\ell_1 \otimes \mathbb{K})^{\otimes n})$$

Let K_n be the space from the proof of Thm. 5.2.7. The map of spectra from (b) translates into a map $\widehat{K}_n \to K_n$, which sends φ to $(\mathrm{id} \otimes \kappa^{\otimes n}) \circ \varphi$. Using the same identification of the homotopy groups with the K-groups as in the proof of Thm. 5.2.7 we have a commutative diagram

$$\begin{tikzcd}
\pi_{i+n}(\widehat{K}_n) \arrow[rr, "\varphi \mapsto (\mathrm{id}\otimes\kappa^{\otimes n})\circ\varphi"] \arrow[dr, "\cong"'] & & \pi_{i+n}(K_n) \arrow[dl, "\cong"] \\
& K_i(C_0(c(X,B), c\,\mathcal{A})) &
\end{tikzcd}$$

proving (c). $\square$

5.3 Twisted K-homology

For twists classified by $H^3(X,\mathbb{Z})$ twisted K-homology is naturally isomorphic to analytic K-homology of a corresponding bundle of compact operators [145]. In this section, we will extend these results to include the higher twists as well. We will start by recalling the definition of topological twisted K-homology from [5, 6], which uses ∞-categories. We will also express twisted K-homology in terms of KK-theory. This is based on a Mayer-Vietoris argument and will not require any knowledge of ∞-categories.

5.3.1 Generalized Thom-spectra

Let R be a ring spectrum. In [5, 6] the authors apply the theory of ∞-categories to associate a Thom spectrum Mf to a map $f\colon X \to BGL_1(R)$, which they use to define twisted versions of R-cohomology and R-homology.

An ∞-category is by definition a simplicial set, which satisfies the weak Kan condition. They were called quasicategories by Joyal. Let $\mathcal{S}$ be the ∞-category of spaces (or ∞-groupoids) and denote by $\mathrm{St}(\mathcal{S})$ its stabilization, i.e. the ∞-category of spectra obtained by taking spectrum objects in $\mathcal{S}$ as described in [95, Sec. 1.4.2]. Denote by $R{-}\mathrm{Mod} \subset \mathrm{St}(\mathcal{S})$ the full subcategory of $\mathrm{St}(\mathcal{S})$ of R-module spectra. Let $R{-}\mathrm{Line} \subset R{-}\mathrm{Mod}$ be the subcategory of free rank one R-modules and equivalences of those. Let $\mathcal{B}\mathrm{Aut}_R(R) \subset R{-}\mathrm{Line}$ be the full subcategory on the object R. The latter are both ∞-groupoids and it was shown in [6, Cor. 2.9] that $\mathcal{B}\mathrm{Aut}_R(R) \simeq R{-}\mathrm{Line}$ and both have the homotopy type $BGL_1(R)$.

To give an ∞-categorical model for $KU^D{-}\mathrm{Line}$ based on C^*-algebras we will use the following technique to turn topological categories into ∞-categories: Given a category $\mathcal{C}$ enriched in topological spaces, we obtain an associated simplicially enriched category $\mathcal{C}^s$ with $\mathrm{Obj}(\mathcal{C}^s) = \mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}_{\mathcal{C}^s}(x,y) = \mathrm{Sing}(\mathrm{Mor}_{\mathcal{C}}(x,y))$, where $\mathrm{Sing}(\,\cdot\,)$ is the singular simplicial set. Let $N_{hc}\colon \mathrm{sCat} \to \mathrm{sSet}$ be the homotopy coherent nerve functor as defined by Cordier [35], [94, Def. 1.1.5.5]. The simplicial sets $\mathrm{Mor}_{\mathcal{C}^s}(x,y)$ are Kan complexes. Therefore $\mathcal{C}^\infty = N_{hc}(\mathcal{C}^s)$ is an ∞-category. Let $\mathcal{B}_D$ be the category with one object and morphism space $\mathrm{Aut}(D\otimes\mathbb{K})$ and let $\mathcal{B}_D^\infty$ be the associated ∞-category. We have a functor $\mathcal{B}_D^\infty \to \mathcal{B}\mathrm{Aut}_{KU^D}(KU_\bullet^{D,\mathrm{mod}}) \subset KU^D{-}\mathrm{Line}$ that sends the object to $KU_\bullet^{D,\mathrm{mod}}$ and maps $\alpha \in \mathrm{Aut}(D\otimes\mathbb{K})$ to the induced map of module spectra as above. On geometric realizations this map agrees with the one from [47, Thm. 4.6], i.e. it induces an isomorphism on π_k for $k>0$ and the inclusion $K_0(D)_+^\times \to K_0(D)^\times$ on π_0.

Definition 5.3.1. Let X be a finite CW-complex, let $\mathcal{P} \to X$ be a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle classified by $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ and let $f^*\mathscr{K}U^D_\bullet \to X$ be the associated bundle of symmetric KU^D-module spectra. Let $\sigma_n\colon X \to f^*\mathscr{K}U^D_n$ be the zero section. The *Thom spectrum* of f is defined by

$$Mf_n = f^*\mathscr{K}U^D_n/\sigma_n(X) = r_! f^*\mathscr{K}U^D_n \ .$$

It is a symmetric module spectrum over $KU^D_\bullet$ with respect to the action maps induced by $\alpha_{\bullet,\bullet}$. Define the topological twisted K-homology of X as

$$K_i^{\mathcal{P},\mathrm{top}}(X) = \pi_i(Mf_\bullet) = \pi_i(r_! f^*\mathscr{K}U^D_\bullet)$$

where $r\colon X \to \{*\}$.

Let $G = \mathrm{Aut}(D \otimes \mathbb{K})$. The spectrum $Mf_\bullet$ in Def. 5.3.1 agrees with the smash product $\Sigma^\infty P_+ \wedge_{\Sigma^\infty G_+} KU^{D,\mathrm{mod}}_\bullet$ in symmetric spectra. Since we assumed X to be a finite CW complex, P has the structure of a (free) G-CW-complex. In particular, $\Sigma^\infty P_+$ turns out to be a cofibrant $\Sigma^\infty G_+$-module spectrum. Therefore the above is equivalent to the derived smash product, which was called the algebraic Thom spectrum in [6]. It was proven in [6, Prop. 3.26] that this notion is equivalent in the stable ∞-category KU^D-Mod to the geometric Thom spectrum [6, Def. 1.4]. Together with [5, Rem. 5.4] we obtain the following:

Theorem 5.3.2. *Let X be a finite CW-complex, let $\mathcal{P} \to X$ be a principal* $\mathrm{Aut}(D \otimes \mathbb{K})$*-bundle pulled back via $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$, let $\mathcal{A}$ be the associated $D \otimes \mathbb{K}$-bundle. Let $\mathcal{P}^- \to X$ be a principal* $\mathrm{Aut}(D \otimes \mathbb{K})$*-bundle, such that the $D \otimes \mathbb{K}$-bundle $\mathcal{A}^-$ associated to it satisfies $C(X, \mathcal{A} \otimes \mathcal{A}^-) \cong C(X, D \otimes \mathbb{K})$. Then we have*

$$K^i_{\mathcal{P}^-}(X) \cong K_i(C(X, \mathcal{A}^-)) \cong \pi_0(\mathrm{Map}_{KU^D}(Mf_\bullet, \Sigma^i KU^{D,\mathrm{mod}}_\bullet) \ .$$

Given a CW-complex X, a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle $\mathcal{P} \to X$ classified by a map $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ and a subcomplex $A \subset X$, there is a canonical map between the associated Thom spectra $M(f|_A)_\bullet \to Mf_\bullet$, which is obtained from the inclusion $(f|_A)^*\mathscr{K}U^D_\bullet \to f^*\mathscr{K}U^D_\bullet$ covering the map $A \to X$.

Theorem 5.3.3. *Let (X, A, B) be an excisive CW-triad. Let $\mathcal{P}$ be a principal* $\mathrm{Aut}(D \otimes \mathbb{K})$*-bundle over X and denote its restrictions to A, B and $A \cap B$ by $\mathcal{P}_A$, $\mathcal{P}_B$ and $\mathcal{P}_{AB}$ respectively. There is a boundary homomorphism $\partial\colon K_i^{\mathcal{P},\mathrm{top}}(X) \to K_{i-1}^{\mathcal{P}_{AB},\mathrm{top}}(A \cap B)$ such that the Mayer-Vietoris sequence of twisted K-homology groups is exact:*

$$\dots \longrightarrow K_i^{\mathcal{P}_{AB},\mathrm{top}}(A \cap B) \longrightarrow K_i^{\mathcal{P}_A,\mathrm{top}}(A) \oplus K_i^{\mathcal{P}_B,\mathrm{top}}(B) \longrightarrow K_i^{\mathcal{P},\mathrm{top}}(X) \xrightarrow{\partial} K_{i-1}^{\mathcal{P}_{AB},\mathrm{top}}(A \cap B) \longrightarrow \dots$$

Proof. Let I be the pushout diagram category and let $\mathcal{S}_{/X}$ be the ∞-category of spaces over X. The pushout diagram $A \longleftarrow A \cap B \longrightarrow B$ defines a functor $F \colon I \to \mathcal{S}_{/X}$, since each of the spaces comes equipped with a map to X. Let $f \colon X \to KU^D{-}\mathrm{Line} \to KU^D{-}\mathrm{Mod}$ be the diagram in the ∞-categorical definition of $Mf_\bullet$. The colimits of $F(i) \to X \to KU^D{-}\mathrm{Mod}$ correspond to $M(f|_{A\cap B})_\bullet$, $M(f|_A)_\bullet$ and $M(f|_B)_\bullet$ respectively. The category I is partially ordered in such a way that F is an order-preserving map from I to the collection of simplicial subsets of X. Using [94, Rem. 4.2.3.9] we see that the conditions of [94, Prop. 4.2.3.8] are satisfied. Thus, [94, Cor. 4.2.3.10] yields that

$$\begin{tikzcd} M(f|_{A\cap B})_\bullet \arrow[r] \arrow[d] & M(f|_A)_\bullet \arrow[d] \\ M(f|_B)_\bullet \arrow[r] & Mf_\bullet \end{tikzcd}$$

is an ∞-categorical pushout diagram in KU^D-Mod. By stability of the latter, this is also a pullback diagram in KU^D-Mod. The associated long exact sequence of homotopy groups is the stated Mayer-Vietoris sequence. □

5.3.2 Analytic twisted K-homology

Let X be a locally compact space, let $\mathcal{P} \to X$ be a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle and denote the associated bundle of C^*-algebras by $\mathcal{A}$. We will use Kasparov's representable bivariant K-theory, which was defined in [84]. Analogous to the twisted K-theory functor, let

$$\begin{aligned} RK^*_{\mathcal{P}}(X) &= \mathcal{R}KK_*(X; C_0(X) \otimes D, C_0(X, \mathcal{A})) \\ RK_*^{\mathcal{P}}(X) &= \mathcal{R}KK_{-*}(X; C_0(X, \mathcal{A}), C_0(X) \otimes D) \\ K_*^{\mathcal{P}}(X, Y) &= KK_{-*}(C_0(X \setminus Y, \mathcal{A}|_{X\setminus Y}), D) \end{aligned}$$

To compare analytic with topological twisted K-homology, we need a version of Poincaré duality. Let M be a smooth compact spin^c manifold with (possibly empty) boundary ∂M. Let $\mathcal{P} \to M$ be a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle with associated bundle of C^*-algebras $\mathcal{A}$. Let $M^\circ = M \setminus \partial M$. The spin^c-condition implies that M has a fundamental class $[M, \partial M] \in K_{\dim(M)}(M, \partial M)$. For a compact space X we have an isomorphism $K^*_{\mathcal{P}}(X) \cong RK^*_{\mathcal{P}}(X)$. Given two continuous $C_0(X)$-algebras A and B the group $\mathcal{R}KK_*(X; A, B)$ maps naturally to $KK_*(A, B)$ by forgetting the additional assumptions on the cycles in representable KK-theory. It was shown in [47, Thm. 3.8] that the isomorphism classes of locally trivial principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundles form an abelian group with respect to the tensor product. Thus, there is a bundle $\mathcal{A}^- \to M$ with the property that $C(M, \mathcal{A}) \otimes_{C(M)} C(M, \mathcal{A}^-) \cong C(M, D \otimes \mathbb{K})$. This yields a class $[\mathcal{A}^-] \in \mathcal{R}KK_0(M; C(M,\mathcal{A}^-), C(M,\mathcal{A}^-))$ and via $x \mapsto x \otimes [\mathcal{A}^-]$ an isomorphism

$$RK^*_{\mathcal{P}}(M) = \mathcal{R}KK_*(M; C(M) \otimes D, C(M,\mathcal{A})) \to \mathcal{R}KK_*(M; C(M,\mathcal{A}^-), C(M) \otimes D)$$

Let $i \colon M^\circ \to M$ be the inclusion and denote by i^* the induced homomorphism on $\mathcal{R}KK_*$. By [57, Prop. 4.9] the homomorphism

$$i^* \colon \mathcal{R}KK_*(M; C(M,\mathcal{A}^-), C(M) \otimes D) \to \mathcal{R}KK_*(M^\circ; C_0(M^\circ,\mathcal{A}^-), C_0(M^\circ) \otimes D)$$

is an isomorphism. Combining the above observations with the map to KK we obtain $\iota_{(M,\partial M)}$:

$$K^*_{\mathcal{P}}(M) \xrightarrow{\cong} \mathcal{R}KK_*(M^\circ; C_0(M^\circ,\mathcal{A}^-), C_0(M^\circ) \otimes D) \longrightarrow KK_*(C_0(M^\circ,\mathcal{A}^-), C_0(M^\circ) \otimes D)\ .$$

We define the Poincaré duality homomorphism by mapping further to $K^{\mathcal{P}-}_{\dim(M)-*}(M,\partial M)$:

$$PD_* \colon K^*_{\mathcal{P}}(M) \to K^{\mathcal{P}-}_{\dim(M)-*}(M,\partial M) \quad ; \quad y \mapsto \iota_{(M,\partial M)}(y) \otimes [M,\partial M]\ . \tag{5.4}$$

The boundary homomorphism $\partial \colon K^{\mathcal{P}-}_{\dim M-*}(M,\partial M) \to K^{\mathcal{P}-}_{\dim(\partial M)-*}(\partial M)$ is given by an intersection product with the class $[\partial] \in KK_1(C(\partial M, \mathcal{A}^-|_{\partial M}), C_0(M^\circ,\mathcal{A}^-))$, which represents the short exact sequence

$$0 \to C_0(M^\circ,\mathcal{A}^-) \to C(M,\mathcal{A}^-) \to C(\partial M, \mathcal{A}^-\big|_{\partial M}) \to 0$$

Lemma 5.3.4. *Let $j \colon \partial M \to M$ be the inclusion, then we have*

$$\partial \otimes \iota_{(M,\partial M)}(x) = (-1)^{\deg(x)}\, \iota_{\partial M}(j^* x) \otimes \partial\ .$$

Proof. There is a second way to obtain the map $\iota_{(M,\partial M)}$, namely via

$$K^*_{\mathcal{P}}(M) \xrightarrow{\otimes[\mathcal{A}^-]} KK_*(C_0(M^\circ,\mathcal{A}^-), C(M,\mathcal{A}) \otimes C_0(M^\circ,\mathcal{A}^-)) \xrightarrow{\mu} KK_*(C_0(M,\mathcal{A}^-), C_0(M^\circ) \otimes D)$$

where μ is induced by the diagonal inclusion $M^\circ \to M \times M^\circ$ and the trivializing isomorphism $C_0(M^\circ, \mathcal{A} \otimes \mathcal{A}^-) \cong C_0(M^\circ, D \otimes \mathbb{K})$. From here the proof proceeds just as in [13, Lem. B.8] $\square$

Corollary 5.3.5. *Let $j \colon \partial M \to M$ be the inclusion, then the following diagram commutes* *up to a sign, which is given by* $(-1)^{\deg(x)}$ *for* $x \in K^*_{\mathcal{P}}(M)$

$$\begin{tikzcd} K^*_{\mathcal{P}}(M) \arrow[r, "j^*"] \arrow[d, "PD_*"'] & K^*_{\mathcal{P}}(\partial M) \arrow[d, "PD_*"] \\ K^{\mathcal{P}-}_{\dim(M)-*}(M, \partial M) \arrow[r, "\partial"'] & K^{\mathcal{P}-}_{\dim(\partial M)-*}(\partial M) \end{tikzcd}$$

Proof. This is a direct consequence of Lemma 5.3.4 and [13, Lem. B.10]. □

Fix a compact Hausdorff space X together with a principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundle $\mathcal{P} \to X$ pulled back via $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$. Let $\widehat{\mathbb{K}}$ be the graded compact operators on a graded separable infinite-dimensional Hilbert space, let

$$\mathscr{K}^D = E\mathrm{Aut}(D \otimes \mathbb{K}) \times_\lambda \hom_{\mathrm{gr}}(\widehat{S}, D \otimes \mathbb{K} \otimes \widehat{\mathbb{K}}) \ ,$$

where λ denotes the left action of $\mathrm{Aut}(D \otimes \mathbb{K})$ on $D \otimes \mathbb{K}$. Let M be a closed spin^c-manifold and let $g\colon M \to f^*\mathscr{K}^D$ be a continuous map to the total space of the bundle. Let $\bar{g}\colon M \to X$ be the map induced by g and the bundle projection. Observe that g yields a class $[E_g] \in K^0_{\bar{g}^*\mathcal{P}}(M)$. There is a homotopy associative multiplication map $m\colon f^*\mathscr{K}^D \wedge \hom_{\mathrm{gr}}(\widehat{S}, \widehat{\mathbb{K}}) \to f^*\mathscr{K}^D$, which uses an isomorphism $\widehat{\mathbb{K}} \otimes \widehat{\mathbb{K}} \cong \widehat{\mathbb{K}}$ of graded C^*-algebras. In particular, the Bott element induces a weak equivalence $\mathscr{K}^D \to \Omega^2\mathscr{K}^D$ (where the loop space is taken fiberwise). The bundle $\mathscr{K}^D$ is the twisted analogue of the space $X \times \mathbb{K}$ in [12, Sec. 8].

Definition 5.3.6. For (M, g) as described above, we define the *analytic index* of the pair by $\mathrm{ind}_a(M, g) = \bar{g}_* PD_0([E_g]) \in K^{\mathcal{P}-}_{\dim(M)}(X)$.

Theorem 5.3.7. *The analytic index only depends on the (spin^c) bordism class* $[M, g] \in \Omega^{\mathrm{spin}^c}(f^*\mathscr{K}^D)$.

Proof. Let (W, h) be a spin^c manifold with boundary $\partial W = M_1 \amalg -M_2$ together with a continuous map $h\colon W \to f^*\mathscr{K}^D$, which restricts to $g_i\colon M_i \to f^*\mathscr{K}^D$. Then we have $PD_0([E_h]) \in K^{\bar{h}^*\mathcal{P}-}_{\dim(W)}(W, \partial W)$. The class $(\bar{h}\big|_{\partial W})_*\partial(PD_0([E_h]))$ vanishes. This follows from the naturality of the boundary map by the commutativity of the following diagram:

$$\begin{tikzcd} K^{\bar{h}^*\mathcal{P}-}_{\dim(W)}(W, \partial W) \arrow[r, "\partial"] \arrow[d, "\bar{h}_*"'] & K^{\bar{h}^*\mathcal{P}-}_{\dim(\partial W)}(\partial W) \arrow[d, "(\bar{h}|_{\partial W})_*"] \\ 0 = K^{\mathcal{P}-}_{\dim(W)}(X, X) \arrow[r, "\partial"'] & K^{\mathcal{P}-}_{\dim(\partial W)}(X) \end{tikzcd}$$

By Cor. 5.3.5, we have $0 = (\bar{h}\big|_{\partial W})_*\partial(PD_0([E_h])) = (\bar{h}\big|_{\partial W})_* PD_0(j^*([E_h])) = \mathrm{ind}_a(M_1, g_1) - \mathrm{ind}_a(M_2, g_2)$, which proves the statement. □

5.3.3 Comparison of topological and analytic twisted K-homology

In this section we prove that the topological twisted K-homology of a finite CW-complex X is isomorphic to its analytic counterpart. To achieve this we extend the idea of [12] to the twisted case, i.e. we construct an intermediate twisted homology theory, which maps to the topological and the analytic one.

As we have seen in the definition of the analytic index map above, the twist changes to its inverse under Poincaré duality. Since this will play a central role in the natural transformations between the theories we are going to construct, we need a functorial way of inverting twists. This is not a problem in [6, 5], where it suffices to have an endofunctor of the ∞-category KU^D-Line. Instead of rephrasing the whole theory in that framework, we simply build the inverse into the category of spaces under consideration.

Let $\mathcal{P}$ and $\mathcal{P}'$ be principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundles over the same base space X. The tensor product of automorphisms induces a group homomorphism $\kappa\colon \mathrm{Aut}(D \otimes \mathbb{K}) \times \mathrm{Aut}(D \otimes \mathbb{K}) \to \mathrm{Aut}((D \otimes \mathbb{K})^{\otimes 2})$ and we define

$$\mathcal{P} \otimes \mathcal{P}' = (\mathcal{P} \times_M \mathcal{P}') \times_\kappa \mathrm{Aut}((D \otimes \mathbb{K})^{\otimes 2})$$

If $\mathcal{A}$ and $\mathcal{A}'$ are the corresponding C^*-algebra bundles, then $\mathcal{A} \otimes \mathcal{A}'$ is associated to $\mathcal{P} \otimes \mathcal{P}'$.

Definition 5.3.8. Let $\mathcal{CW}_D^{\mathrm{fin}}$ be the following category: The objects are tuples (X, f, f^-, τ), where X is a finite CW-complex, $f, f^-\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ are continuous maps classifying bundles $\mathcal{P}_f = f^*E\mathrm{Aut}(D \otimes \mathbb{K})$ and $\mathcal{P}_f^- = (f^-)^*E\mathrm{Aut}(D \otimes \mathbb{K})$ and $\tau\colon \mathcal{P}_f \otimes \mathcal{P}_f^- \to X \times \mathrm{Aut}((D \otimes \mathbb{K})^{\otimes 2})$ is a trivialization. A morphism $(X, f, f^-, \tau) \to (X', f', f'^-, \tau')$ is tuple $(\varphi, \hat{\varphi}, \hat{\varphi}^-, \rho)$, where $\varphi\colon X \to X'$ is a continuous map and $\hat{\varphi}\colon \mathcal{P}_f \to \mathcal{P}_{f'}$, $\hat{\varphi}^-\colon \mathcal{P}_f^- \to \mathcal{P}_{f'}^-$ are maps of principal $\mathrm{Aut}(D \otimes \mathbb{K})$-bundles covering φ and $\rho\colon X \to \mathrm{Aut}((D \otimes \mathbb{K})^{\otimes 2})$ is a change of trivialization, which fits into the commutative diagram

$$\begin{tikzcd}
\mathcal{P}_f \otimes \mathcal{P}_f^- \arrow[r, "\tau"] \arrow[d, "\hat{\varphi} \otimes \hat{\varphi}^-"'] & X \times \mathrm{Aut}((D \otimes \mathbb{K})^{\otimes 2}) \arrow[d, "\lambda_\rho"] \\
\mathcal{P}_{f'} \otimes \mathcal{P}_{f'}^- \arrow[r, "\tau'"] & X \times \mathrm{Aut}((D \otimes \mathbb{K})^{\otimes 2})
\end{tikzcd}$$

where λ denotes the left multiplication. If the bundles are clear, we will sometimes abbreviate the tuple (X, f, f^-, τ) by X and a morphism $(\varphi, \hat{\varphi}, \hat{\varphi}^-, \rho)$ by φ. Let $X, Y \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ and let $\varphi, \varphi'\colon X \to Y$ be two morphisms. A *homotopy* between φ and φ' is a homotopy $H\colon X \times I \to Y$ between φ and φ' that preserves all the further structure in the sense that H is covered by homotopies of principal bundle maps $\hat{H}\colon \mathcal{P}_f \times I \to \mathcal{P}_{f'}$ and $\hat{H}^-\colon \mathcal{P}_f^- \times I \to \mathcal{P}_{f'}^-$ and a corresponding homotopy of the changes of trivializations.

Remark 5.3.9. Given a CW-complex X, an object $Y \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ and a map $\varphi\colon X \to Y$ all extra data can be pulled back to X along φ. In particular, a subcomplex $A \subset X$ can be extended to an object $A \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$.

If $\varphi\colon X \to Y$ as above is a homotopy equivalence with homotopy inverse $\psi\colon Y \to X$, then we can find maps of principal bundles $\hat{\varphi}\colon \varphi^*\mathcal{P}_f \to \mathcal{P}_f$ and $\hat{\psi}\colon \mathcal{P}_f \to \varphi^*\mathcal{P}_f$ covering φ and ψ respectively. This can be done in such a way that $\hat{\varphi} \circ \hat{\psi}$ and $\hat{\psi} \circ \hat{\varphi}$ are both homotopic to automorphisms of principal bundles covering the identity. A corresponding argument for the trivializations yields the following: Given an object $(Y, f, f^-, \tau) \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$, a CW-complex X and a homotopy equivalence $\varphi\colon X \to Y$, there are morphisms $X \to Y$ and $Y \to X$ in $\mathcal{CW}_D^{\mathrm{fin}}$, such that both compositions are homotopic to automorphisms of $(X, f\circ\varphi, f^-\circ\varphi, \tau\circ\varphi)$ and (Y, f, f^-, τ), respectively. In particular, any homotopy invariant functor will map $(\varphi, \hat{\varphi}, \hat{\varphi}^-, \rho)$ to an isomorphism.

Theorem 5.3.10. *Let $h_*, h'_*\colon \mathcal{CW}_D^{\mathrm{fin}} \to \mathrm{GrAb}$ be two functors to the category of graded abelian groups with the following properties:*

(i) homotopy invariance*: If $\varphi, \varphi'\colon X \to Y$ are morphisms such that φ is homotopic to φ' in the sense of Def. 5.3.8, then $h_k(\varphi) = h_k(\varphi')$.*

(ii) Mayer-Vietoris sequence*: If $X \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ and $A \subset X$, $B \subset X$ are subcomplexes with $X = A \cup B$. Then there is a natural transformation $\partial\colon h_*(X) \to h_{*-1}(A\cap B)$ and the following sequence, in which the unlabeled arrows are induced by inclusions as in the ordinary Mayer-Vietoris sequence, is exact:*

$$\longrightarrow h_k(A\cap B) \longrightarrow h_k(A) \oplus h_k(B) \longrightarrow h_k(X) \xrightarrow{\partial} h_{k-1}(A\cap B) \longrightarrow$$

Let $\eta\colon h_* \to h'_*$ be a natural transformation. If $\eta_*\colon h(*) \to h'(*)$ is an isomorphism of graded abelian groups for all objects $* \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ that have the one-point space as underlying CW-complex. Then η_X is an isomorphism for all objects $X \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$.

Proof. Let $X \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$. The underlying CW-complex of X is homotopy equivalent to a finite simplicial complex. After barycentric subdivision there exists a cover by subcomplexes $(U_i)_{i\in J}$, such that for every $k \in \mathbb{N}$ each intersection $U_{i_1} \cap \dots \cap U_{i_k}$ is either contractible or empty. (Such a cover will be called *good* in the following.) Thus, we obtain a CW-complex X', which has a good cover, and a homotopy equivalence $\varphi\colon X \to X'$ with homotopy inverse $\psi\colon X' \to X$. Using the pullback along ψ, we obtain $X' \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$. The composition $\varphi\circ\psi$ is homotopic to an automorphism of $X' \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ and likewise for $\psi\circ\varphi$. Therefore $h(\varphi)$ is an isomorphism and it suffices to show the statement for objects, which allow good covers.

Suppose that we have shown that η_Y is an isomorphism for those $Y \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$, which have a good cover by n subcomplexes. This holds for $n = 1$ by homotopy invariance and the assumption on η. Let $Y' \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ be an object which has a good cover by subcomplexes $V_1, \dots, V_{n+1}$, let $V = V_1 \cup \dots \cup V_n$ and note that V and $V \cap V_{n+1}$ satisfy the induction hypothesis. The five lemma applied to the following comparison diagram of Mayer-Vietoris sequences

$$\begin{tikzcd}
{} \arrow[r] & h_k(V) \oplus h_k(V_{n+1}) \arrow[r] \arrow[d, "\cong"] & h_k(V \cup V_{n+1}) \arrow[r, "\partial"] \arrow[d] & h_{k-1}(V \cap V_{n+1}) \arrow[r] \arrow[d, "\cong"'] & {} \\
{} \arrow[r] & h'_k(V) \oplus h'_k(V_{n+1}) \arrow[r] & h'_k(V \cup V_{n+1}) \arrow[r, "\partial"] & h'_{k-1}(V \cap V_{n+1}) \arrow[r] & {}
\end{tikzcd}$$

proves the induction step and therefore the statement. $\square$

The intermediate twisted homology theory alluded to in the introduction is defined as follows: Let $(X, f, f^-, \tau) \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ and consider the framed bordism group $\Omega_*^{\mathrm{fr}}((f^-)^*\mathscr{K}^D)$. Given $(M, g\colon M \to (f^-)^*\mathscr{K}^D) \in \Omega_{n+2k}^{\mathrm{fr}}((f^-)^*\mathscr{K}^D)$, we obtain an element $(M \times S^2, g')$ (where the stable framing on S^2 is the one induced by $S^2 \subset \mathbb{R}^3$) in $\Omega_{n+2(k+1)}^{\mathrm{fr}}((f^-)^*\mathscr{K}^D)$, where g' is given by

$$g' \colon M \times S^2 \xrightarrow{(g,b)} (f^-)^*\mathscr{K}^D \times \hom_{\mathrm{gr}}(\widehat{S}, \widehat{\mathbb{K}}) \xrightarrow{\ m\ } (f^-)^*\mathscr{K}^D$$

and $b\colon S^2 \to \hom_{\mathrm{gr}}(\widehat{S}, \widehat{\mathbb{K}})$ represents the Bott class.

Definition 5.3.11. Let $(X, f, f^-, \tau) \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$, let $\mathcal{P}^- = \mathcal{P}_f^-$ and define

$$k_n^{\mathcal{P}^-}(X) = \lim_k \Omega_{n+2k}^{\mathrm{fr}}((f^-)^*\mathscr{K}^D) \ .$$

The direct limit is formed with respect to the maps

$$\Omega_{n+2k}^{\mathrm{fr}}((f^-)^*\mathscr{K}U_2^D) \to \Omega_{n+2(k+1)}^{\mathrm{fr}}((f^-)^*\mathscr{K}U_2^D)$$

described above. This is a functor $\mathcal{CW}_D^{\mathrm{fin}} \to \mathrm{GrAb}$.

Lemma 5.3.12. *The functor $X \mapsto k_n^{\mathcal{P}^-}(X)$ satisfies the conditions of Theorem 5.3.10.*

Proof. Let $(X, f, f^-, \tau), (Y, g, g^-, \tau') \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$. A homotopy $H\colon X \times I \to Y$ is by definition covered by principal bundle maps $\mathcal{P}_f^- \times I \to \mathcal{P}_g^-$ inducing a homotopy $(f^-)^*\mathscr{K}^D \times I \to (g^-)^*\mathscr{K}^D$. Thus, homotopy invariance follows from the homotopy invariance of framed bordism.

Let $X \in \mathrm{Obj}(\mathcal{CW}_D^{\mathrm{fin}})$ and let A, B be subcomplexes, such that $X = A \cup B$. The double mapping cylinder $c(A, A \cap B, B) = (A \amalg (A \cap B) \times I \amalg B)/\sim$ is homotopy equivalent to X via $\theta\colon c(A, A \cap B, B) \to X$. It can be covered by open sets U and V, such that $(c(A, A\cap B, B), U, V)$ is an excisive triad, U is homotopy equivalent to A, V to B and $U\cap V$ to $A\cap B$. All equivalences are induced by θ. Pulling back the bundle $\mathcal{P}^-$ to $c(A, A\cap B, B)$ yields an excisive cover of $\theta^*(f^-)^*\mathscr{K}^D$ by $\theta^*(f^-)^*\mathscr{K}^D\big|_U$ and $\theta^*(f^-)^*\mathscr{K}^D\big|_V$. The observation in Rem. 5.3.9 yields that $\Omega^{\mathrm{fr}}_{n+2k}(\theta^*(f^-)^*\mathscr{K}^D\big|_U) \to \Omega^{\mathrm{fr}}_{n+2k}((f^-)^*\mathscr{K}^D\big|_A)$ is an isomorphism and likewise for the spaces over $A\cap B$, B and X. Since direct limits preserve exact sequences, we therefore obtain the Mayer-Vietoris sequence for $k_*^{\mathcal{P}^-}$ from the one for framed bordism. □

Definition 5.3.13. Let $KU_n^{D,\mathrm{mod},\mathrm{gr}} = \hom_{\mathrm{gr}}(\widehat{S}, D\otimes\mathbb{K}\otimes\widehat{\mathbb{K}}\otimes(\mathbb{C}\ell_1\otimes D\otimes\mathbb{K})^{\otimes n})$. Analogous to $KU_\bullet^{D,\mathrm{mod}}$ there also are action maps $\alpha_{\bullet,\bullet}$ such that $(KU_\bullet^{D,\mathrm{mod},\mathrm{gr}}, \alpha_{\bullet,\bullet})$ is a symmetric module spectrum over $KU_\bullet^D$ with the same properties as $KU_\bullet^{D,\mathrm{mod}}$ by a similar proof as for Thm. 5.2.4. Let $\mathscr{K}U_n^{D,\mathrm{gr}} = E\mathrm{Aut}(D \otimes \mathbb{K}) \times_\lambda KU_n^{D,\mathrm{mod},\mathrm{gr}}$, where the left action acts on the $D \otimes \mathbb{K}$-factor of the tensor product. The *graded Thom spectrum* of a map $f\colon X \to B\mathrm{Aut}(D \otimes \mathbb{K})$ is defined by

$$Mf_n^{\mathrm{gr}} = r_! f^* \mathscr{K}U_n^{D,\mathrm{gr}} \ .$$

It is a symmetric module spectrum over $KU_\bullet^D$ with respect to the action maps induced by $\alpha_{\bullet,\bullet}$. Let $\mathcal{P} = f^*E\mathrm{Aut}(D \otimes \mathbb{K})$. We define $K_n^{\mathcal{P},\mathrm{top},\mathrm{gr}}(X) = \pi_n(Mf_\bullet^{\mathrm{gr}})$.

Let $e \in \widehat{\mathbb{K}}$ be a rank 1-projection in the even part of the algebra. We have a C^*-algebra homomorphism $D \otimes \mathbb{K} \to D \otimes \mathbb{K} \otimes \widehat{\mathbb{K}}$ given by $a \mapsto a \otimes e$. Since this stabilization induces an isomorphism on K-groups, we obtain a π_*-equivalence of spectra $KU_\bullet^{D,\mathrm{mod}} \to KU_\bullet^{D,\mathrm{mod},\mathrm{gr}}$ and a corresponding map of symmetric $KU_\bullet^D$-module spectra $Mf_\bullet \to Mf_\bullet^{\mathrm{gr}}$.

Lemma 5.3.14. *The map $Mf_\bullet \to Mf_\bullet^{\mathrm{gr}}$ induces a natural isomorphism of the associated homology theories $K_\bullet^{\mathcal{P},\mathrm{top}} \to K_\bullet^{\mathcal{P},\mathrm{top},\mathrm{gr}}$. In particular, it is a π_*-equivalence and therefore a stable one.*

Proof. By Thm. 5.3.3 the functors $K_\bullet^{\mathcal{P},\mathrm{top}}$ and $K_\bullet^{\mathcal{P},\mathrm{top},\mathrm{gr}}$ on $\mathcal{CW}_D^{\mathrm{fin}}$ satisfy the conditions of Thm. 5.3.10. Therefore it suffices to check that $K_n^{\mathcal{P},\mathrm{top}}(pt) \to K_n^{\mathcal{P},\mathrm{top},\mathrm{gr}}(pt)$ is an isomorphism. But over a point we have $Mf_\bullet = KU_\bullet^{D,\mathrm{mod}}$ and $Mf_\bullet^{\mathrm{gr}} = KU_\bullet^{D,\mathrm{mod},\mathrm{gr}}$ and the statement follows from the observation above. □

Multiplication by the Bott element $[b] \in K_2(pt)$ induces an isomorphism of twisted K-homology groups $KK_{-n}(C(X,\mathcal{A}), D) \to KK_{-(n+2)}(C(X,\mathcal{A}), D)$. The element $[b]$ can be represented by a pointed map $S^2 \to \hom_{\mathrm{gr}}(\widehat{S}, \widehat{\mathbb{K}})$. Using the same construction as for $\mathscr{K}^D$, there is an action map

$$Mf_n^{\mathrm{gr}} \wedge \hom_{\mathrm{gr}}(\widehat{S}, \widehat{\mathbb{K}}) \to Mf_n^{\mathrm{gr}} \ .$$

Just as for $\mathscr{K}^D$ this yields a Bott map $Mf_n^{\mathrm{gr}} \to \Omega^2 Mf_n^{\mathrm{gr}}$, which induces

$$K_n^{\mathcal{P},\mathrm{top},\mathrm{gr}}(X) = \pi_n(Mf_\bullet^{\mathrm{gr}}) \to \pi_n(\Omega^2 Mf_\bullet^{\mathrm{gr}}) \cong \pi_{n+2}(Mf_\bullet^{\mathrm{gr}}) = K_{n+2}^{\mathcal{P},\mathrm{top},\mathrm{gr}}(X)\ .$$

Definition 5.3.15. We define

$$\begin{aligned} K_n^{\mathcal{P},\mathrm{an},\lim}(X) &= \lim_k KK_{-(n+2k)}(C(X,\mathcal{A}),D) \\ K_n^{\mathcal{P}^-,\mathrm{top},\lim}(X) &= \lim_k K_{n+2k}^{\mathcal{P}^-,\mathrm{top},\mathrm{gr}}(X) \end{aligned}$$

where the direct limit in both cases is taken over the Bott homomorphisms described above.

These are both functors $\mathcal{CW}_D^{\mathrm{fin}} \to \mathrm{GrAb}$. It is immediate that $K_n^{\mathcal{P},\mathrm{an}}(X) \to K_n^{\mathcal{P},\mathrm{an},\lim}(X)$ is an isomorphism. The same holds true for $K_n^{\mathcal{P}^-,\mathrm{top},\mathrm{gr}}(X) \to K_n^{\mathcal{P}^-,\mathrm{top},\lim}(X)$ by Thm. 5.3.10 and the fact that the Bott homomorphism induces an isomorphism $K_n^{\mathcal{P}^-,\mathrm{top},\mathrm{gr}}(pt) \to K_{n+2}^{\mathcal{P}^-,\mathrm{top},\mathrm{gr}}(pt)$ on coefficients.

Theorem 5.3.16. *Let D be a strongly self-absorbing C^*-algebra, which satisfies the UCT. The analytic index map induces a natural isomorphism of functors*

$$\mathrm{ind}_a \colon k_n^{\mathcal{P}^-} \to K_n^{\mathcal{P},\mathrm{an},\lim}\ .$$

Proof. The Poincaré duality homomorphism (5.4) required the choice of isomorphisms

$$C(X,\mathcal{A}\otimes\mathcal{A}^-) \cong C(X,(D\otimes\mathbb{K})^{\otimes 2}) \cong C(X,D\otimes\mathbb{K})$$

Since $\mathcal{P}$, its inverse $\mathcal{P}^-$ and the trivialization $\tau\colon \mathcal{P}\otimes\mathcal{P}^- \to X \times \mathrm{Aut}((D\otimes\mathbb{K})^{\otimes 2})$ are part of the objects in $\mathcal{CW}_D^{\mathrm{fin}}$, the first isomorphism is canonical. For the second identification there is a canonical path connected space of choices given by pairs of isomorphisms $D\otimes D\cong D$ and $\mathbb{K}\otimes\mathbb{K}\cong\mathbb{K}$ [47, Thm. 2.3]. Moreover, a stable framing of a manifold M determines a spin^c structure on it. By Thm. 5.3.7 we therefore obtain a natural transformation

$$\Omega_n^{\mathrm{fr}}((f^-)^*\mathscr{K}^D) \to K_n^{\mathcal{P},\mathrm{an}}(X) \quad ; \quad [M,g] \mapsto \mathrm{ind}_a(M,g)\ .$$

To check that this construction is compatible with direct limits over the Bott isomorphisms note that $[E_{m\circ(g\times\eta_2)}] = [E_g] \boxtimes [B] \in K_{\mathcal{P}}^0(M\times S^2)$, where $[B]\in K_0(S^2)$ is the Bott bundle (i.e. the Hopf bundle minus the trivial line bundle) and the tensor product is induced by the exterior tensor product. Let $q\colon S^2 \to \{pt\}$ denote the map to the point. The product with

$q_*(\iota_{S^2}([B]) \otimes [S^2]) = [b] \in K_2(pt)$ is the Bott isomorphism. We have

$$\begin{aligned}\mathrm{ind}_a(M \times S^2, m \circ (g \times \eta_2)) &= (\bar{g}_* \circ \mathrm{pr}_{M*}) PD_0([E_{m\circ(g\times\eta_2)}]) \\ &= (\bar{g}_* \circ \mathrm{pr}_{M*}) \big((\iota_M([E_g]) \otimes [M]) \boxtimes (\iota_{S^2}([B]) \otimes [S^2])\big) \\ &= \bar{g}_* PD_0([E_g]) \cdot [b] = \mathrm{ind}_a(M, g) \cdot [b]\end{aligned}$$

Therefore we obtain a natural transformation $k_n^{\mathcal{P}^-} \to K_n^{\mathcal{P},\mathrm{an},\lim}$. Since the right hand side is homotopy invariant and has Mayer-Vietoris sequences by [18, Thm. 21.5.1], the conditions of Thm. 5.3.10 are satisfied and it remains to be checked that $k_n^{\mathcal{P}^-}(pt) \to K_n^{\mathcal{P},\mathrm{an},\lim}(pt)$ is an isomorphism, where $\mathcal{P} = \mathrm{Aut}(D \otimes \mathbb{K})$ is the trivial bundle over pt. By the Pontrjagin-Thom construction the n-th framed cobordism group $\Omega_n^{\mathrm{fr}}(W)$ of an unbased space W agrees with the stable homotopy group $\pi_n^s(W_+)$, where W_+ is W with a disjoint basepoint. Let $K^D = \hom_{\mathrm{gr}}(\widehat{S}, D \otimes \mathbb{K} \otimes \widehat{\mathbb{K}})$. Using the same argument as in [12, Sec. 9.1] (see the remark below) it follows that

$$k_n^{\mathcal{P}}(pt) = \lim_k \Omega_{n+2k}^{\mathrm{fr}}(K^D) \cong \lim_k \pi_{n+2k}^s((K^D)_+) \cong \pi_n(K^D) \ .$$

The unit map induces an isomorphism $KK_i(D,D) \cong K_i(D)$ by [51, Thm. 3.4]. The homomorphism $\pi_0(K^D) \to k_0^{\mathcal{P}}(pt) \to K_0^{\mathcal{P},\mathrm{an},\lim}(pt) \cong KK(D,D) \cong K_0(D)$ sends a point in K^D to the K-theory class it represents, possibly altered by an automorphism of $D \otimes \mathbb{K}$ that arises from the trivialization of $\mathcal{P} \otimes \mathcal{P}^- \to pt$. This is an isomorphism. Since we assumed that D satisfies the UCT, we have $KK_1(D,D) \cong K_1(D) \cong 0$ by [139, Prop. 5.1], which implies that $0 = k_1^{\mathcal{P}}(pt) \to K_1^{\mathcal{P},\mathrm{an},\lim}(pt) = 0$ is an isomorphism. □

Remark 5.3.17. The basepoint is missing in the argument given in [12, Sec. 9.1], i.e. it should be $\pi_{n+2k}(S^{2m} \wedge \mathbb{K}_+)$ in the square diagram on page 21. The calculation of the homotopy groups is nevertheless correct. This follows from the fact that the map $b\colon S^2 \wedge \mathbb{K}_+ \to \mathbb{K}_+$ induced by Bott periodicity factors through $\mathbb{K}$ (note that $\mathbb{K}$ already has a basepoint). The induced homomorphism $\pi_{n+2k}(S^{2m} \wedge \mathbb{K}_+) \to \pi_{n+2k+2}(S^{2m} \wedge \mathbb{K})$ yields an isomorphism of the direct limits. We obtain $\lim_k \pi_{n+2k}^s((K^D)_+) \cong \pi_n(K^D)$ in the same way.

Observe that there are canonical maps $\mu\colon ((f^-)^*\mathscr{K}^D)_+ \wedge KU_n^D \to r_!(f^-)^*\mathscr{K}U_n^{D,\mathrm{gr}} = Mf_n^{-,\mathrm{gr}}$, which yield a corresponding map of symmetric spectra. Let $\mathbb{S}_\bullet$ be the sphere spectrum and consider $\mathbb{S}_\bullet \wedge ((f^-)^*\mathscr{K}^D)_+ \to Mf_\bullet^{-,\mathrm{gr}}$ given by

$$S^n \wedge ((f^-)^*\mathscr{K}^D)_+ \xrightarrow{\eta_n \wedge \mathrm{id}} KU_n^D \wedge ((f^-)^*\mathscr{K}^D)_+ \xrightarrow{\mu\circ\tau} Mf_n^{-,\mathrm{gr}}$$

where τ just switches the factors in the wedge product. By the Pontryagin-Thom construction we have $\Omega_n^{\mathrm{fr}}((f^-)^*\mathscr{K}^D) \cong \pi_n^s(((f^-)^*\mathscr{K}^D)_+) \cong \pi_n(\mathbb{S}_\bullet \wedge ((f^-)^*\mathscr{K}^D)_+)$ and the above map induces a transformation $\Omega^{\mathrm{fr}}_n((f^-)^*\mathscr{K}^D) \to \pi_n(Mf^{-,\mathrm{gr}}_\bullet) = K^{\mathcal{P}^-,\mathrm{top},\mathrm{gr}}_n(X)$. The limits on both sides were designed in such a way that this extends to a natural transformation

$$k^{\mathcal{P}^-}_n \to K^{\mathcal{P}^-,\mathrm{top},\mathrm{lim}}_n \ .$$

Theorem 5.3.18. *The above transformation is a natural isomorphism $k^{\mathcal{P}^-}_n \to K^{\mathcal{P}^-,\mathrm{top},\mathrm{lim}}_n$. If D is a strongly self-absorbing C^*-algebra that satisfies the UCT, we obtain a natural isomorphism of functors $\mathcal{CW}^{\mathrm{fin}}_D \to \mathrm{GrAb}$ of the form*

$$K^{\mathcal{P}^-,\mathrm{top}}_\bullet \cong K^{\mathcal{P},\mathrm{an}}_\bullet \ .$$

Proof. We have seen in Lem. 5.3.12 and Thm. 5.3.3 that the functors $k^{\mathcal{P}^-}_\bullet$ and $K^{\mathcal{P}^-,\mathrm{top},\mathrm{lim}}_\bullet$ satisfy the conditions of Thm. 5.3.10. To prove the first statement, it therefore suffices to check that $k^{\mathcal{P}^-}_n(pt) \to K^{\mathcal{P}^-,\mathrm{top},\mathrm{lim}}_n(pt)$ is an isomorphism. In the case $X = pt$ we have $Mf^{-,\mathrm{gr}}_\bullet = KU^{D,\mathrm{mod},\mathrm{gr}}_\bullet$, in particular $Mf^{-,\mathrm{gr}}_0 = K^D$ in the notation of the proof of Thm. 5.3.16. As already stated there, we have

$$k^{\mathcal{P}^-}_n(pt) \cong \lim_k \pi^s_{n+2k}((K^D)_+) \cong \pi_n(K^D)$$

and the following square commutes

$$\begin{tikzcd}
\pi_n(K^D) \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & \pi_n(Mf^{-,\mathrm{gr}}_0) \arrow[d, "\cong"] \\
k^{\mathcal{P}^-}_n(pt) \cong \lim_k \pi^s_{n+2k}((K^D)_+) \arrow[rr] & & \lim_k \pi_{n+2k}(Mf^{-,\mathrm{gr}}_\bullet) \cong K^{\mathcal{P}^-,\mathrm{top},\mathrm{lim}}(pt)
\end{tikzcd}$$

proving the first claim. For the second statement we combine all natural isomorphisms

$$K^{\mathcal{P}^-,\mathrm{top}}_\bullet \xrightarrow{\cong} K^{\mathcal{P}^-,\mathrm{top},\mathrm{gr}}_\bullet \xrightarrow{\cong} K^{\mathcal{P}^-,\mathrm{top},\mathrm{lim}}_\bullet \xleftarrow{\cong} k^{\mathcal{P}^-}_\bullet \xrightarrow{\cong} K^{\mathcal{P},\mathrm{an},\mathrm{lim}}_\bullet \xleftarrow{\cong} K^{\mathcal{P},\mathrm{an}}_\bullet$$

proven above. □

Remark 5.3.19. It is conjectured that every separable nuclear C^*-algebra satisfies the universal coefficient theorem (UCT) in KK-theory needed above and all the known examples of strongly self-absorbing C^*-algebras do.

5.4 Higher twisted K-theory of suspensions

Let D be a strongly self-absorbing C^*-algebra, which satisfies the universal coeffient theorem. In this section we will construct a noncommutative model of all twists of K-theory for suspensions and compute the corresponding twisted K-groups, in particular those of the odd-dimensional spheres. Observe that $gl_1(KU^D)^1(S^{2k}) = [S^{2k}, BGL_1(KU^D)] = 0$, therefore the twisted K-groups of even-dimensional spheres agree with the untwisted ones.

5.4.1 Morita equivalences and linking algebras

Given two C^*-algebras A and B, an *A-B-(Morita) equivalence* is a full right Hilbert B-module X together with an isomorphism $A \to \mathcal{K}_B(X)$ onto the compact B-linear operators on X. Equivalently, X is a Banach space that is a full left Hilbert A-module as well as a full right Hilbert B-module, such that the inner products satisfy ${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$. If X is an A-B-equivalence, then the adjoint module X^* is a B-A-equivalence. This is the inverse equivalence in the sense that the inner products induce canonical bimodule isomorphisms $X \otimes_B X^* \cong A$ and $X^* \otimes_A X \cong B$. We refer the reader to [115, 25] for a detailed exposition. We can associate to an A-B-equivalence X the *linking algebra* $L(X)$, which contains A and B as complementary full corners [25, Thm. 1.1]. This is given by

$$L(X) = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

with the multiplication and $*$-operation:

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + {}_A\langle \xi_1, \eta_2 \rangle & a_1 \xi_2 + \xi_1 b_2 \\ \eta_1^* a_2 + b_1 \eta_2^* & \langle \eta_1, \xi_2 \rangle_B + b_1 b_2 \end{pmatrix} \quad , \quad \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix} .$$

If A and B are unital, then X is finitely generated and projective as a right Hilbert B- and a left Hilbert A-module. In this case we have an induced isomorphism $X_* \colon K_0(A) \to K_0(B)$, which sends a finitely generated projective right Hilbert A-module E to $E \otimes_A X$.

Let $H_B = \ell^2(\mathbb{N}) \otimes B$. If we choose an isomorphism of right Hilbert B-modules $X \cong pH_B$ for a projection $p \in B \otimes \mathbb{K}$, then the left action of A on X identifies A with the corner $p(B \otimes \mathbb{K})p \subset B \otimes \mathbb{K}$. In particular, X induces an injective $*$-homomorphism $\iota_X \colon A \otimes \mathbb{K} \to B \otimes \mathbb{K} \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Let $q \in A \otimes \mathbb{K}$ be a projection with $E \cong qH_A$, then $X_*(E)$ corresponds to the projection $\iota_X(q)$.

Let $p_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in L(X)$ and $p_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in L(X)$. $Y_0 = p_0 L(X)$ is a Morita equivalence between $A = p_0 L(X) p_0$ and $L(X)$. Similarly, $Y_1 = p_1 L(X)$ is a B-$L(X)$-equivalence. The following diagram of isomorphisms commutes:

$$\begin{tikzcd} K_0(A) \arrow[rr, "X_*"] \arrow[dr, "Y_{0*}"'] & & K_0(B) \arrow[dl, "Y_{1*}"] \\ & K_0(L(X)) & \end{tikzcd}$$

As above, Y_{0*} agrees with the map that sends a projection $q \in A \otimes \mathbb{K}$ to $\left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right) \in L(X) \otimes \mathbb{K}$. Apart from Y_{0*} we have another natural homomorphism $K_0(A) \to K_0(L(X))$ induced by a stable isomorphism $\varphi_0 \colon A \otimes \mathbb{K} \to L(X) \otimes \mathbb{K}$ constructed as follows: Consider $p_0 \otimes 1 \in M(L(X) \otimes \mathbb{K})$. By [24, Lem. 2.5] there exists a partial isometry $v \in M(L(X) \otimes \mathbb{K})$ with $v^*v = 1$ and $vv^* = p_0 \otimes 1$, which induces φ_0 via $\varphi_0(a) = v^* \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) v$. Analogously, we define the isomorphism $\varphi_1 \colon B \otimes \mathbb{K} \to L(X) \otimes \mathbb{K}$.

Lemma 5.4.1. *The isomorphism $\varphi_{0*} \colon K_0(A) \to K_0(L(X))$ does not depend on the choice of v, v^* with the above properties. Moreover, we have $Y_{i*} = \varphi_{i*}$ for $i \in \{0,1\}$ and $(\varphi_1^{-1} \circ \varphi_0)_* = X_*$.*

Proof. If $w \in M(L(X) \otimes \mathbb{K})$ is another partial isometry with $w^*w = 1$ and $ww^* = p_0 \otimes 1$, then $u = v^*w$ is a unitary with $vu = w$. Since $U(M(L(X) \otimes \mathbb{K}))$ is path-connected [37], the two homomorphisms $K_0(A) \to K_0(L(X))$ corresponding to v and w agree.

Let $\iota \colon A \otimes \mathbb{K} \to L(X) \otimes \mathbb{K}$ be given by $\iota(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By the above remarks we have $\iota_* = Y_{0*}$. Moreover, $\iota \circ \varphi_0^{-1}(x) = vxv^*$. We will construct a strictly continuous path $v_t \in M(L(X) \otimes \mathbb{K})$ for $t \in [0,1]$ with $v_0 = v$ and $v_1 = 1$. This will prove the second claim. Let $H_L = \ell^2(\mathbb{N}) \otimes L(X)$. We can identify $M(L(X) \otimes \mathbb{K})$ with $\mathcal{L}_{L(X)}(H_L)$, i.e. the bounded adjointable operators on H_L. There are two notions of strict topology on both algebras, which agree on the unit ball [88, Prop. 8.1].

Choose an isomorphism $H_L \oplus H_L \to H_L$ and denote by r_0, $r_1 \colon H_L \to H_L$ the precomposition of this isomorphism with the natural maps $H_L \to H_L \oplus H_L$. In particular, we have $r_i^* r_j = \delta_{ij} 1$. Using the same idea as in the proof of [17, Thm. I.3.2.10] we obtain a strictly continuous path of partial isometries $w_t \colon H_L \to H_L$ with $w_t^* w_t = 1$, $w_1 = r_0$ and $w_0 = 1$. Denote the corresponding path for r_1 by w_t''. Now consider $w_t' = \sqrt{1-t}\, r_0 v + \sqrt{t}\, r_1$, which is a strictly continuous path of partial isometries with $(w_t')^* w_t' = 1$, $w_0' = r_0 v$ and $w_1' = r_1$. v_t is now given by concatenating the paths $w_t\, v$, w_t' and w_{1-t}''. ☐

5.4.2 Construction of higher twists

Let $X = \Sigma Y$ be the suspension of a finite-dimensional compact metrizable space Y. We will assume that Y is cofibrantly pointed so that the map from the unreduced to the reduced suspension is a homotopy equivalence. Choose contractible closed subsets $W_0, W_1 \subset X$, contractible open subsets $U_i \subset W_i$ and contractible closed subsets $V_i \subset U_i$, such that each of the pairs W_0, W_1, U_0, U_1 and V_0, V_1 covers X and such that $W_0 \cap W_1 \simeq Y$, $U_0 \cap U_1 \simeq Y$ and $V_0 \cap V_1 \simeq Y$. The twists of the K-theory of X are given by

$$gl_1(KU^D)^1(\Sigma Y) = [Y, \Omega B\mathrm{Aut}(D \otimes \mathbb{K})] = [Y, \mathrm{Aut}(D \otimes \mathbb{K})] \cong K_0(C(Y) \otimes D)^{\times}_{+} \ .$$

In the following we will make this isomorphism precise by constructing a continuous $C(X)$-algebra A with fibers stably isomorphic to D, which satisfies the generalized Fell condition [47, Def. 4.1]. By [47, Thm. 4.2], $A \otimes \mathbb{K}$ is then isomorphic to the section algebra of a locally trivial continuous bundle with fiber $D \otimes \mathbb{K}$.

Let $B = C(X) \otimes D \otimes \mathbb{K}$, $B_0 = C_0(U_1 \cap U_2) \otimes D \otimes \mathbb{K}$ and let $p \in B(W_0 \cap W_1)$ be a full projection representing an invertible element $[p] \in K_0(B(W_0 \cap W_1))^{\times}_{+}$. By [50, Thm. 0.1] and [47, Lem. 2.14] the continuous $C(W_0 \cap W_1)$-algebra $pB(W_0 \cap W_1)p$ is isomorphic to $C(W_0 \cap W_1) \otimes D$. Let $q_0 = 1_{C(X)} \otimes 1_D \otimes e \in B$ and let $\varphi\colon C(W_0 \cap W_1) \otimes D \to pB(W_0 \cap W_1)p$ be an isomorphism, which turns $H' = pB(W_0 \cap W_1)q_0$ into a self-Morita equivalence of $C(W_0 \cap W_1) \otimes D$. Let $H_0' = pB_0q_0 \subset H'$. Since φ restricts to an isomorphism $C_0(U_0 \cap U_1) \otimes D \to pB_0p$, H_0' is a self-Morita equivalence of $C_0(U_0 \cap U_1) \otimes D$. We can extend H_0' to a (non-full) Hilbert $C(X) \otimes D$-bimodule H by defining the multiplication with functions in $C(X) \otimes D$ via restriction. The inner product takes values in $C_0(U_0 \cap U_1) \otimes D$, which sits inside $C(X) \otimes D$ via extension by 0. Note that the fiber $H(x) = H/C_0(X \setminus \{x\})H$ is zero if $x \notin U_0 \cap U_1$. Let H^* be the analogous extension of $(H_0')^*$. Let $A_0 = C_0(U_0) \otimes D$ and extend it to a continuous $C(X)$-algebra in the same way, likewise define $A_1 = C_0(U_1) \otimes D$. Now consider the linking algebra of H given by

$$A = \begin{pmatrix} A_0 & H \\ H^* & A_1 \end{pmatrix}$$

Lemma 5.4.2. *The C^*-algebra A is a continuous $C(X)$-algebra. All fibers of $A \otimes \mathbb{K}$ are isomorphic to $D \otimes \mathbb{K}$ and $A \otimes \mathbb{K}$ satisfies the generalized Fell condition. Therefore there is a locally trivial bundle of C^*-algebras $\mathcal{A} \to X$ with fiber $D \otimes \mathbb{K}$, such that $C(X, \mathcal{A}) \cong A$ as $C(X)$-algebras.*

Proof. By construction A is a continuous $C(X)$-algebra. The statement about the fibers is clear for all $x \in X \setminus U_i$. For $x \in U_0 \cap U_1$ the fiber is given by the stabilization of the linking algebra $L(H(x)) = \left(\begin{smallmatrix} D & H(x) \\ H(x)^* & D \end{smallmatrix}\right)$. But this is also isomorphic to $D \otimes \mathbb{K}$. Consider the closed cover of X by V_0, V_1 and note that $A(V_0) \otimes \mathbb{K}$ contains the projection $p_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes e$, likewise we have $p_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes e \in A(V_1) \otimes \mathbb{K}$. Both projections represent invertible elements in $K_0(A(x))$. Thus, A satisfies the generalized Fell condition. The existence of $\mathcal{A}$ now follows from [47, Thm. 4.2]. □

Lemma 5.4.3. *Let $\mathcal{A} \to X$ be the locally trivial bundle from the last lemma. The isomorphism $\kappa\colon gl_1(KU^D)^1(X) = [X, B\mathrm{Aut}(D \otimes \mathbb{K})] \to K_0(C(Y) \otimes D)^{\times}_{+}$ maps $[\mathcal{A}]$ to $[p]$.*

Proof. Let $\phi\colon V_0 \cap V_1 \to \mathrm{Aut}(D \otimes \mathbb{K})$ be the transition function of $\mathcal{A} \to X$ corresponding to a choice of trivialization of $\mathcal{A}$ over V_0 and V_1. Since X is a suspension, ϕ completely determines the isomorphism class of $\mathcal{A}$. It induces a $*$-homomorphism $\hat{\phi}\colon D \otimes \mathbb{K} \to C(V_0 \cap V_1) \otimes D \otimes \mathbb{K}$. By definition of κ we have $\kappa([\mathcal{A}]) = [\hat{\phi}(1 \otimes e)] \in K_0(C(V_0 \cap V_1) \otimes D)^{\times}_{+}$ and the class does not depend on the trivializations. Choose an isomorphism $C(X, \mathcal{A}) \to A \otimes \mathbb{K}$. In particular, we obtain trivializations $C(V_i, \mathcal{A}|_{V_i}) \to A(V_i) \otimes \mathbb{K} \to C(V_i) \otimes D \otimes \mathbb{K}$ using the projections $p_i \in A(V_i) \otimes \mathbb{K}$ to construct partial isometries as described in the paragraph preceding Lemma 5.4.1. Using the fact that $A(V_0 \cap V_1) \otimes \mathbb{K} = L(H(V_0 \cap V_1)) \otimes \mathbb{K}$ it follows from this lemma that the homomorphism $\hat{\phi}$ corresponding to those trivializations maps $1 \otimes e$ to the right Hilbert $C(V_0 \cap V_1) \otimes D$-module $H(V_0 \cap V_1) = p\, H_{C(V_0 \cap V_1) \otimes D}$, which is precisely the one represented by the projection p. $\square$

Remark 5.4.4. Let $G^{(1)} = \amalg_{i,j} U_i \cap U_j$, $G^{(0)} = \amalg_i U_i$ with $i, j \in \{0, 1\}$. These form the morphism and object space of the Čech groupoid associated to the open cover $U_0 \cup U_1 = X$. Let $F \to G^{(1)}$ be the Fell bundle given by $F_{00} = U_0 \times D$ over $U_0 \cap U_0$, $F_{11} = U_1 \times D$ over $U_1 \cap U_1$,

$$F_{01} = \{(x, \xi) \in U_0 \cap U_1 \times H_D \mid p(x)\xi = \xi\}$$

over $U_0 \cap U_1$ and $F_{10} = F_{01}^{*}$ (applied fiberwise) over $U_1 \cap U_0$. The multiplication of this Fell bundle is either given by the multiplication in the algebra or by left (respectively right) multiplication on the bimodule bundle. The above algebra A corresponds to the C^*-algebra $C^*(F)$ associated to this Fell bundle. This point of view allows us to generalize the construction to spaces which are not suspensions.

Observe that $K_0(C(Y) \otimes D)$ is a right module over $K_0(D)$. Given $[p] \in K_0(C(Y) \otimes D)$ and $[q] \in K_0(D)$, we will denote the action by $[p] \cdot [q]$. The reduced K-theory $\widetilde{K}_0(C(Y) \otimes D)$ of a compact space Y with coefficients in D is defined to be the cokernel of the map $K_0(D) \to K_0(C(Y) \otimes D)$ induced by the unital homomorphism $d \mapsto 1_{C(Y)} \otimes d$.

Theorem 5.4.5. *Let D be a strongly self-absorbing C^*-algebra that satisfies the UCT. Let $\mathcal{A} \to X$ be the bundle of C^*-algebras associated to a projection $p \in C(Y) \otimes D \otimes \mathbb{K}$ by the construction described above. Let G be the additive subgroup given by $G = \{x \in K_0(D) \mid [p] \cdot x = [1] \cdot y \in K_0(C(Y) \otimes D)$ for some $y \in K_0(D)\}$. Then we have a short exact sequence*

$$0 \to K_1(C(Y) \otimes D) \to K^0_{\mathcal{P}}(X) \to G \to 0\ .$$

where the surjection is induced by evaluation at a point $x_0 \in X$. Moreover, $K^1_{\mathcal{P}}(X) \cong \widetilde{K}_0(C(Y) \otimes D)/([p])$.

Proof. Note that $C(V_0 \cap V_1, \mathcal{A}|_{V_0\cap V_1}) \cong A(V_0\cap V_1) \otimes \mathbb{K} \cong C(V_0 \cap V_1) \otimes D \otimes \mathbb{K}$. Since V_0 and V_1 are contractible, $K_1(D) = 0$ and $K_0(C(V_0 \cap V_1) \otimes D) \cong K_0(C(Y) \otimes D)$, the Mayer-Vietoris six-term exact sequence of X boils down to

$$\begin{tikzcd} K_0(C(X,\mathcal{A})) \arrow[r] & K_0(D) \oplus K_0(D) \arrow[r, "\theta"] & K_0(C(Y) \otimes D) \arrow[d] \\ K_1(C(Y) \otimes D) \arrow[u] & 0 \arrow[l] & K_1(C(X,\mathcal{A})) \arrow[l] \end{tikzcd}$$

We have $K_0(C(V_0 \cap V_1) \otimes D) \cong K_0(C(Y) \otimes D)$ and we can identify $C(V_0 \cap V_1, \mathcal{A}|_{V_0\cap V_1})$ with $A(V_0 \cap V_1) \otimes \mathbb{K} = L(H(V_0 \cap V_1)) \otimes \mathbb{K}$. With the notation from the paragraph preceding Lemma 5.4.1 we obtain as a consequence of that lemma that

$$\theta(x, y) = \varphi_{1*}^{-1}(Y_{0*}(x) - Y_{1*}(y)) = H_*(x) - [1] \cdot y = [p] \cdot x - [1] \cdot y \ . \tag{5.5}$$

Thus, $\ker(\theta) \cong G$, which implies the existence of the claimed short exact sequence for $K_0(C(X,\mathcal{A}))$. The statement about $K_1(C(X,\mathcal{A}))$ is clear from (5.5). $\square$

Let $[H] \in K^0(S^{2k})$ be the class represented by the canonical line bundle H over $\mathbb{C}P^1$. We have $K^1(S^{2k}) = 0$ and $K^0(S^{2k}) \cong \mathbb{Z}[t]/(t^2)$, where $t = [H] - [1]$. Since D satisfies the UCT, we have $K_0(C(S^{2k}) \otimes D) \cong K_0(D)[t]/(t^2)$. Invertible elements in this ring are of the form $a + bt$ with $a, b \in K_0(D)$ and a invertible.

Corollary 5.4.6. *Let D be a strongly self-absorbing C^*-algebra that satisfies the UCT. Let $\mathcal{A} \to S^{2k+1}$ be a locally trivial bundle with fiber $D \otimes \mathbb{K}$ associated to a projection $p \in C(S^{2k}) \otimes D \otimes \mathbb{K}$ as described above. Define $a, b \in K_0(D)$ by $[p] = a + bt \in K_0(C(S^{2k}) \otimes D)$ using the isomorphism from the last paragraph and suppose that $b \neq 0$. Then*

$$K^0_{\mathcal{P}}(S^{2k+1}) = 0 \ ,$$
$$K^1_{\mathcal{P}}(S^{2k+1}) \cong K_0(D)/(b) \ .$$

Proof. If $b \neq 0$, the group G from Theorem 5.4.5 is trivial. Moreover, $K_1(C(S^{2k}) \otimes D) = 0$. Thus, $K^0_{\mathcal{P}}(S^{2k+1}) = 0$. The reduced K-group $\widetilde{K}_0(C(S^{2k}) \otimes D)$ is isomorphic to $K_0(D)$ and the quotient map $K_0(C(S^{2k}) \otimes D) \to \widetilde{K}_0(C(S^{2k}) \otimes D) \cong K_0(D)$ can be identified with $(x + yt) \mapsto y$. This implies the statement about $K^1_{\mathcal{P}}(S^{2k+1})$. $\square$

6 Deformations of nilpotent groups and homotopy symmetric C^*-algebras

The homotopy symmetric C^*-algebras are those separable C^*-algebras for which one can unsuspend in E-theory. We find a new simple condition that characterizes homotopy symmetric nuclear C^*-algebras and use it to show that the property of being homotopy symmetric passes to nuclear C^*-subalgebras and it has a number of other significant permanence properties.

As an application, we show that if $I(G)$ is the kernel of the trivial representation $\iota\colon C^*(G) \to \mathbb{C}$ for a countable discrete torsion free nilpotent group G, then $I(G)$ is homotopy symmetric and hence the Kasparov group $KK(I(G), B)$ can be realized as the homotopy classes of asymptotic morphisms $[[I(G), B \otimes \mathcal{K}]]$ for any separable C^*-algebra B.

6.1 Introduction

The intuitive idea of deformations of C^*-algebras was formalized by Connes and Higson who introduced the concept of asymptotic morphism [34]. These morphisms are at the heart of E-theory, the universal bifunctor from the category of separable C^*-algebras to the category abelian groups which is homotopy invariant, C^*-stable and half-exact. Asymptotic morphisms have become important tools in other areas, such as deformation quantization [116], index theory [32], [140], the Baum-Connes conjecture [70], shape theory [40], and classification theory of nuclear C^*-algebras [119].

An asymptotic morphism $(\varphi_t)_{t\in[0,\infty)}$ is given by a family of maps $\varphi_t\colon A \to B$ parametrized by $t \in [0,\infty)$ such that $t \mapsto \varphi_t$ is pointwise continuous and the axioms for $*$-homomorphisms are satisfied *asymptotically* for $t \to \infty$. There is a natural notion of homotopy based on asymptotic morphisms of the form $A \to B \otimes C[0,1]$. Homotopy classes of asymptotic morphisms from the suspension SA of A to the stabilization of SB provide a model for $E(A,B)$, i.e. $E(A,B) = [[SA, SB \otimes \mathbb{K}]]$, [34]. A similar construction using completely positive contractive asymptotic morphisms yields a realization of KK-Theory, namely $KK(A,B) \cong [[SA, SB \otimes \mathbb{K}]]^{\mathrm{cp}}$, as proven by Houghton-Larsen and Thomsen [73]. E-theory factors through KK-theory and the fact that the map $KK(A,B) \to E(A,B)$ is an isomorphism for nuclear C^*-algebras A can be easily seen from the Choi-Effros theorem, which implies $[[A,B]] \cong [[A,B]]^{\mathrm{cp}}$.

Note that the introduction of the suspensions and the stabilization of B are necessary to obtain a natural abelian group structure on $E(A,B)$. Equally important, but perhaps less apparent, is the fact that SA becomes quasidiagonal, as shown by Voiculescu [144], and this property implicitly assures a large supply of almost multiplicative maps $SA \to \mathbb{K}$. However, a deformation $A \to B \otimes \mathbb{K}$, without the suspensions, contains in principle more geometric information. Asymptotic morphisms of this form are a crucial tool in the classification theory of nuclear C^*-algebras. We are confronted with the dilemma of understanding $[[A, B \otimes \mathbb{K}]]$, while only $E(A,B) = [[SA, SB \otimes \mathbb{K}]]$ is computable using the tools of algebraic topology. The best case scenario in this situation is of course that the monoid homomorphism $[[A, B \otimes \mathbb{K}]] \to E(A,B)$ induced by the suspension map is an isomorphism.

Under favorable circumstances, this is in fact true: It is shown in [45] and [38] that for a connected compact metrizable space X with basepoint $x_0 \in X$, we have $[[C_0(X \setminus x_0), B \otimes \mathbb{K}]] \cong E(C_0(X \setminus x_0), B)$. In particular, $K_0(X \setminus x_0) \cong [[C_0(X \setminus x_0), \mathbb{K}]]$. On the right hand side we can replace the compact operators $\mathbb{K}$ by $\bigcup_{n=1}^{\infty} M_n(\mathbb{C})$. Thus, the reduced K-homology of X classifies deformations of $C_0(X \setminus x_0)$ into matrices. Similar deformations of commutative C^*-algebras into matrix algebras appeared in condensed matter physics, where Kitaev [85] proposed a classification of topological insulators via (real) K-homology. We refer the reader to the recent work of Loring [92] for further developments and additional references.

A full answer to the question of unsuspending in E-theory was found in [45]: The natural map $[[A, B \otimes \mathbb{K}]] \to E(A,B)$ is an isomorphism for all separable C^*-algebras B if and only if A is *homotopy symmetric*, which means that $[[\mathrm{id}_A]] \in [[A, A \otimes \mathbb{K}]]$ has an additive inverse or equivalently that $[[A, A \otimes \mathbb{K}]]$ is a group.

As nice as this condition is, it can be quite hard to check in practice. One of the two main results in this paper is Theorem 6.3.1, which shows that being homotopy symmetric is equivalent for separable nuclear C^*-algebras to a property which is significantly easier to verify. The proof of this theorem relies crucially on results of Thomsen [136]. The new property, which we call property (QH), see Definition 6.2.6 (i), makes sense for all separable C^*-algebras and has the important feature that it passes to C^*-subalgebras. This allows us to exhibit new vast classes of homotopy symmetric C^*-algebras, see Theorem 6.3.3 and Corollary 6.3.4.

Our second main result is Theorem 6.4.3 which states that the augmentation ideal $I(G) = \ker(\iota : C^*(G) \to \mathbb{C})$ for a torsion free countable discrete nilpotent group G satisfies property (QH) and hence it is homotopy symmetric. In particular $[[I(G), B \otimes \mathbb{K}]] \cong KK(I(G), B)$ for any separable C^*-algebra B. This confirms a conjecture of the first author [44] for the class of nilpotent groups.

6.2 Discrete asymptotic morphisms and Property (QH)

Definition 6.2.1. Let A, $(B_n)_n$ be separable C^*-algebras. A completely positive contractive (cpc) *discrete asymptotic morphism* from A to $(B_n)_n$ is a sequence of completely positive linear contractions $\{\varphi_n : A \to B_n\}_n$ such that

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$$

for all $a, b \in A$. If, furthemore, each φ_n is unital we say (φ_n) is a *ucp discrete asymptotic morphism*. Two cpc discrete asymptotic morphisms $\{\varphi_n, \psi_n : A \to B_n\}_n$ are *equivalent*, written $\varphi_n \approx \psi_n$, if $\lim_{n\to\infty} \|\varphi_n(a) - \psi_n(a)\| = 0$ for all $a \in A$. They are *unitarily equivalent*, written $(\varphi_n) \sim (\psi_n)$, if there is a sequence of unitaries $u_n \in B_n$ such that $(\varphi_n) \approx (u_n\psi_n u_n^*)$. They are *homotopy equivalent*, written $(\varphi_n) \sim_h (\psi_n)$, if there is a cpc discrete asymptotic morphism $\{\Phi_n\colon A \to C[0,1] \otimes B_n\}_n$ such that $\Phi_n^{(0)} = \varphi_n$ and $\Phi_n^{(1)} = \psi_n$ for all $n \geq 1$. Equivalent discrete asymptotic morphisms are homotopic via the homotopy $\Phi_n^{(t)} = (1 - t)\varphi_n + t\psi_n$. Homotopy of ucp asymptotic morphisms is defined similarly, by requiring that $\Phi_n(1) = 1$. The homotopy classes of cpc discrete asymptotic morphisms $\varphi_n\colon A \to B$ from A to a fixed $B = B_n$ will be denoted by $[[A, B]]_{\mathbb{N}}^{\mathrm{cp}}$. (Non-discrete) *cpc asymptotic morphisms* are defined completely analogously by replacing the indexing set $\mathbb{N}$ by $[0, \infty)$ and considering cpc maps $\varphi\colon A \to C_b([0, \infty), B)$. The homotopy classes of cpc asymptotic morphisms are denoted by $[[A, B]]^{\mathrm{cp}}$. Similarly we denote by $[[A, B]]$ (respectively $[[A, B]]_{\mathbb{N}}$ in the discrete case) the homotopy classes of general asymptotic morphisms, [34], [136].

Remark 6.2.2. If A and B are separable C^*-algebras, then every cpc discrete asymptotic morphism $\{\varphi_n\colon A \to B \otimes \mathbb{K}\}_n$ is equivalent to some $\{\psi_n\colon A \to M_{m(n)}(B)\}_n$ obtained by compressing φ_n by a suitable sequence of projections $1_{M(B)} \otimes p_n \in M(B) \otimes \mathbb{K}$, where $p_n \in \mathbb{K}$ converges strongly to 1.

Remark 6.2.3. There is a category *Asym* with objects separable C^*-algebras and with homotopy classes of (non-discrete) asymptotic morphisms as morphisms. In particular, there is a composition of asymptotic morphisms, which is well-defined up to homotopy [34]. There is a similar category $\mathit{Asym}^{\mathrm{cp}}$ based on cpc asymptotic morphisms. There are no analogues of these categories for homotopy classes of discrete asymptotic morphisms. Nevertheless, it was shown in [136, Thm. 7.2] that there is a well-defined pairing of the form $[[A, B]]_{\mathbb{N}} \times [[B, C]] \to [[A, C]]_{\mathbb{N}}$, $(x, y) \mapsto y \circ x$ such that $z \circ (y \circ x) = (z \circ y) \circ x$ for $z \in [[C, D]]$. The arguments of [136, Thm. 7.2] show that there is also a pairing $[[A, B]]_{\mathbb{N}}^{\mathrm{cp}} \times [[B, C]]^{\mathrm{cp}} \to [[A, C]]_{\mathbb{N}}^{\mathrm{cp}}$ with similar properties.

Any discrete asymptotic morphism $\{\varphi_n : A \to B_n\}_n$ induces a $*$-homomorphism $\mathbf{\Phi}$: $A \to \prod_n B_n / \bigoplus_n B_n$.

Let us recall that by a result of Voiculescu [144], a separable C^*-algebra is quasidiagonal if there is an injective $*$-homomorphism $\eta : A \to \prod_n \mathbb{K}/\bigoplus_n \mathbb{K}$ which is liftable to a cpc map $\eta : A \to \prod_n \mathbb{K}$.

Definition 6.2.4. A discrete asymptotic morphism $\{\varphi_n : A \to B_n\}_n$ is called *injective* if the induced $*$-homomorphism $\mathbf{\Phi}$ is injective, or equivalently if
$\limsup_n \|\varphi_n(a)\| = \|a\|$ for all $a \in A$.

The cone over a C^*-algebra B is defined as $CB = C_0(0,1] \otimes B$.

Proposition 6.2.5. *For a separable C^*-algebra A the following properties are equivalent.*

(i) There is a null-homotopic injective *cpc discrete asymptotic morphism $\{\eta_n\colon A \to \mathbb{K}\}_n$.*

(ii) There is a null-homotopic injective *cpc discrete asymptotic morphism $\{\gamma_n\colon A \to L(H)\}_n$.*

(iii) There is an injective $*$-homomorphism $\eta : A \to \prod_n C\mathbb{K}/\bigoplus_n C\mathbb{K}$ which is liftable to a cpc map $\eta : A \to \prod_n C\mathbb{K}$.

(iv) There is an injective $*$-homomorphism $\gamma : A \to \prod_n CL(H)/\bigoplus_n CL(H)$ which is liftable to a cpc map $\gamma : A \to \prod_n CL(H)$.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious since $\mathbb{K} \subset L(H)$. The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are are also straightforward.

(iv) $\Rightarrow$ (i) Let γ be as in (iv) with components γ_n. Since γ is injective, we must have $\limsup \|\gamma_n(a)\| = \|a\|$ for all $a \in A$. Let $(a_i)_i$ be a dense sequence in A. For each i there is a strictly increasing sequence $(m(i,n))_n$ such that $\|\gamma_{m(i,n)}(a_i)\| \geq \|a_i\| - 1/n$, for all $n \geq 1$. If we define $\gamma'_n = \gamma_{m(1,n)} \oplus \gamma_{m(2,n)} \oplus \cdots \oplus \gamma_{m(n,n)}$, then $\lim \|\gamma'_n(a)\| = \|a\|$ for all $a \in A$. Therefore we may assume that (γ_n) had this property in the first place.

Since A is separable, there is a separable C^*-algebra $D \subset L(H)$ such that $\gamma_n(A) \subset D$ for all $n \geq 1$. Consider an injective $*$-homomorphism $j : C_0(0,1] \to C_0(0,1] \otimes C_0(0,1]$, for example $j(f)(s,t) = f(\min(s,t))$. Since $C_0(0,1] \otimes D$ is quasidiagonal by [144], there is a cpc asymptotic morphism $\{\theta_n : C_0(0,1] \otimes D \to \mathbb{K}\}_n$ with $\lim_n \|\theta_n(b)\| = \|b\|$ for all b. After passing to a subsequence $(\theta_n)_n$ we may arrange that the maps η_n obtained as the compositions

$$A \xrightarrow{\gamma_n} C_0(0,1] \otimes D \xrightarrow{j \otimes \mathrm{id}_D} C_0(0,1] \otimes C_0(0,1] \otimes D \xrightarrow{\mathrm{id} \otimes \theta_n} C_0(0,1] \otimes \mathbb{K}$$

define an asymptotic morphism

$$\{\eta_n : A \to C_0(0,1] \otimes \mathbb{K}\}_n$$

such that $\lim_n \|\eta_n(a)\| = \|a\|$ for all $a \in A$.

We regard $\eta_n : A \to C\mathbb{K}$ as a continuous family of maps $\{\eta_n^{(t)} : A \to \mathbb{K}\}_{t\in[0,1]}$ with $\eta_n^{(0)} = 0$. Next we want to arrange that $\lim_n \|\eta_n^{(1)}(a)\| = \|a\|$ for all $a \in A$. Since $\lim_n \|\eta_n(a)\| = \|a\|$ for all $a \in A$, after passing to a subsequence of $(\eta_n)_n$, we may arrange for each $n \geq 1$, $\|\eta_n(a_i)\| > \|a_i\| - 1/i$ for all $1 \leq i \leq n$. Therefore for every n and i with $1 \leq i \leq n$, there is $t_{n,i} \in [0,1]$ such that $\|\eta_n^{(t_{n,i})}(a_i)\| > \|a_i\| - 1/i$. Define $E_n : A \to C\mathbb{K}$ by $E_n^{(t)} = \bigoplus_{i=1}^n \eta_n^{(tt_{n,i})}$ for $t \in [0,1]$. It follows immediately that $\{E_n : A \to C\mathbb{K}\}_n$ is a cpc asymptotic morphism such that $\lim_n \|E_n^{(1)}(a)\| = \|a\|$ for all $a \in A$. □

Definition 6.2.6. (i) A separable C^*-algebra A has *property (QH)* if it satisfies one of the equivalent conditions from Proposition 6.2.5.

(ii) For a countable discrete group G, denote the character induced by the trivial representation by $\iota\colon C^*(G) \to \mathbb{C}$ and set $I(G) = \ker(\iota)$. We say that G has *property (QH)* if $I(G)$ has property (QH). One can reformulate this condition as follows. There is a discrete injective ucp asymptotic morphism $\{\pi_n : C^*(G) \to M_{m(n)}(\mathbb{C})\}_n$ which is homotopic to $\{\iota_n : C^*(G) \to M_{m(n)}(\mathbb{C})\}_n$, where each ι_n is the multiple $m(n)\cdot\iota$ of the trivial representation ι.

Example 6.2.7. If X is a compact connected metrizable space and $x_0 \in X$, then $C_0(X \setminus x_0)$ has property (QH). In particular, if G is a torsion free countable discrete abelian group, then $I(G) \cong C_0(\widehat{G} \setminus \iota)$ has property (QH).

Remark 6.2.8. Let us note that if A has property (QH), then A must be quasidiagonal. This follows from a result of Voiculescu [144] which states that contractible C^*-algebras are quasidiagonal. Moreover A cannot have any nonzero projections since $CL(H)$ does not contain nonzero projections. In particular, if a discrete group G has property (QH), then G must be torsion free, see Remark 6.4.4.

It is easily verified that

(i) Property (QH) passes to C^*-subalgebras.

(ii) If a separable C^*-algebra A has property (QH) then so does $A \otimes_{min} B$ for any separable C^*-algebra B.

Other permanence properties are proved in Theorem 6.3.3.

The following Lemma is crucial for the proof that property (QH) implies the existence of additive inverses of homotopy classes of discrete asymptotic morphisms. It is a slight generalization of [41, Lem. 5.3]. The proof is almost identical, except that one has to replace Voiculescu's theorem and Stinespring's theorem by Kasparov's generalization of these [83]. We will use the notation from [41, Sec. 5]: If A and B are C^*-algebras, E, F are right Hilbert

B-modules, $\mathcal{F} \subset A$ is a finite set, $\epsilon > 0$ and $\varphi\colon A \to \mathcal{L}_B(E)$ and $\psi\colon A \to \mathcal{L}_B(F)$ are two maps, we write $\varphi \prec_{\mathcal{F},\epsilon} \psi$ if there is an isometry $v \in \mathcal{L}_B(E,F)$ such that $\|\varphi(a) - v^*\psi(a)v\| < \epsilon$ for all $a \in \mathcal{F}$. If v can be chosen to be a unitary, we write $\varphi \sim_{\mathcal{F},\epsilon} \psi$. Moreover, we write $\varphi \prec \psi$ if $\varphi \prec_{\mathcal{F},\epsilon} \psi$ for all finite sets $\mathcal{F}$ and for all $\epsilon > 0$.

Lemma 6.2.9. *Let A and B be separable unital C^*-algebras such that A or B is nuclear. Let $\{\varphi_n\colon A \to M_{k(n)}(B)\}_n$ and $\{\gamma_n\colon A \to M_{r(n)}(\mathbb{C})\}_n$ be ucp discrete asymptotic morphisms. Suppose that (γ_n) is injective. Then there exist a sequence $(\omega(n))$ of disjoint finite subsets of $\mathbb{N}$ with $\max\omega(n-1) < \min\omega(n)$ and a ucp discrete asymptotic morphism $\{\varphi'_n\colon A \to M_{s(n)}(B)\}_n$ such that $(\varphi_n \oplus \varphi'_n) \sim (\gamma_{\omega(n)} \otimes 1_B)$, where $\gamma_{\omega(n)} = \oplus_{i\in\omega(n)}\gamma_i$.*

Proof. There exists a sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ of finite selfadjoint subsets of A consisting of unitaries such that their union is dense in $U(A)$ and a sequence $\epsilon_1 \geq \epsilon_2 \geq \dots$ convergent to zero such that φ_n is $(\mathcal{F}_n, \epsilon_n)$-multiplicative. By [41, Lem. 5.1] it suffices to construct a sequence $(\omega(n))$ such that $\varphi_n \prec_{\mathcal{F}_n,\epsilon_n} \gamma_{\omega(n)}$.

We will construct $(\omega(n))$ inductively. Suppose that we already have $\omega(1),\dots,\omega(n-1)$ and choose $m > \max\omega(n-1)$ such that

$$\Gamma = \oplus_{i\geq m}\gamma_i\colon A \to \prod_{i\geq m} M_{r(i)}(\mathbb{C}) \subset \mathcal{L}(H)$$

is $(\mathcal{F}_n, \frac{\epsilon_n^2}{4})$-multiplicative. Observe that $\Gamma\colon A \to \mathcal{L}(H)$ is a unital $*$-monomorphism modulo the compact operators. Let $\pi\colon A \to \mathcal{L}(H)$ be a faithful unital representation with $\pi(A) \cap \mathcal{K}(H) = \{0\}$, then we have $\Gamma \sim_{\mathcal{F}_n,\epsilon_n} \pi$ by [26, Lem. 2.1] (or [41, Lem. 5.2]). Note that there is a natural embedding $\mathcal{L}(H) \to \mathcal{L}_B(H_B)$, which sends T to $T \otimes 1_B$. Moreover, $\Gamma \otimes 1_B \sim_{\mathcal{F}_n,\epsilon_n} \pi \otimes 1_B$. Let π_n be the Stinespring dilation of φ_n obtained via [83, Thm. 3]. We have that $\pi_n \oplus (\pi \otimes 1_B) \sim (\pi \otimes 1_B)$ by [83, Thm. 6]. Thus,

$$\varphi_n \prec \pi_n \prec \pi_n \oplus (\pi \otimes 1_B) \sim (\pi \otimes 1_B) \sim_{\mathcal{F}_n,\epsilon_n} \Gamma \otimes 1_B\ ,$$

hence $\varphi_n \prec_{\mathcal{F}_n,\epsilon_n} \Gamma \otimes 1_B$. Let $v \in \mathcal{L}_B(B^{k(n)}, H_B)$ be a partial isometry such that $\|\varphi_n(a) - v^*(\Gamma \otimes 1_B)(a)v\| < \epsilon_n$ for all $a \in \mathcal{F}_n$. The projections of $M_\infty(B)$ are dense in those of $B \otimes \mathcal{K}(H)$. Thus, we can find an isometry $w \in \mathcal{L}_B(B^{k(n)}, B^N)$ which approximates v in norm sufficiently well so that $\|\varphi_n(a) - w^*(\Gamma \otimes 1_B)(a)w\| < \epsilon_n$ for all $a \in \mathcal{F}_n$. Here we identify $B^N \subset H_B$ as the submodule of the first N coordinates. It follows that if we let $\omega(n) = \{m, m+1, \dots, m+N\}$, then $\varphi_n \prec_{\mathcal{F}_n,\epsilon_n} \gamma_{\omega(n)}$. ∎

Lemma 6.2.10. *Let $\theta : M \to G$ be a morphism of monoids with $\theta(M) = G$. If G is a group and all the elements of $\theta^{-1}(1_G)$ are invertible in M, then M is a group.*

Proof. For $x \in M$, choose $x' \in M$ such that $\theta(x') = \theta(x)^{-1}$. Then $\theta(xx') = \theta(x'x) = 1_G$. It follows that both xx' and $x'x$ are invertible and hence there are $y_1, y_2 \in M$ such that $x(x'y_1) = 1_M = (y_2x')x$. Thus x is invertible. $\square$

Proposition 6.2.11. *Let A, B be separable C^*-algebras such that either A or B is nuclear. If A has property (QH), then $[[A, B \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is a group.*

Proof. We will first prove the proposition for unital C^*-algebras B. Let $\{\varphi_n \colon A \to M_{r(n)}(B)\}_n$ be a discrete cpc asymptotic morphism. By Remark 6.2.2 it suffices to construct an additive inverse of $[[\varphi_n]]$. Since A has property (QH), there exists an injective cpc discrete asymptotic morphism $\{\eta_n \colon A \to M_{s(n)}(\mathbb{C})\}_n$ which is null-homotopic.

Let $\widetilde{A}$ denote the unitization of A and set $R(n) = r(n) + 1$. Let $\widetilde{\varphi}_n \colon \widetilde{A} \to M_{R(n)}(B)$ be the unital extension of φ_n, so that $\widetilde{\varphi}_n(1) = 1_{M_{R(n)}} \otimes 1_B$. This is a ucp asymptotic morphism. Likewise, let $\widetilde{\eta}_n \colon \widetilde{A} \to M_{S(n)}(\mathbb{C})$ for $S(n) = s(n) + 1$ be the unitization of η_n. Note that $(\widetilde{\eta}_n)$ is still injective. From Lemma 6.2.9 we obtain sequences $(\omega(n))$ and $\{\widetilde{\varphi}'_n \colon A \to M_{T(n)}(B)\}_n$ such that $(\widetilde{\varphi}_n \oplus \widetilde{\varphi}'_n) \sim (\widetilde{\eta}_{\omega(n)} \otimes 1_B)$. Let $j \colon A \to \widetilde{A}$ be the inclusion map and set $\varphi'_n = \widetilde{\varphi}'_n \circ j$. Then $(\varphi_n \oplus \varphi'_n) \sim (\eta_{\omega(n)} \otimes 1_B)$ which is null-homotopic.

Now consider the case when B is nonunital. Observe that for any short exact sequence of separable C^*-algebras

$$0 \to I \to B \xrightarrow{p} D \to 0 \tag{6.1}$$

and an arbitrary separable C^*-algebra A there is a corresponding long exact Puppe sequence of pointed sets:

$$[[A, SB]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, SD]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, C_p]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, B]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, D]]_{\mathbb{N}}^{\mathrm{cp}} \tag{6.2}$$

where $C_p = \{(b, f) \in B \oplus C_0([0,1), D) \mid f(0) = p(b)\}$ is the mapping cone of the $*$-homomorphism p. The proof of exactness is entirely similar to the proof for the Puppe sequence in E-theory, see [39, Prop. 6]. Indeed as shown in the proof of [121, Thm. 3.8], the mapping cone of the map $C_p \to B$, $(b, f) \mapsto b$ is homotopic as a C^*-algebra to SD.

If the short exact sequence (6.1) splits, the first and the last map in (6.2) are surjective and we obtain the short exact sequence

$$0 \longrightarrow [[A, C_p]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, B]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, D]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow 0.$$

Moreover, it was proven in [45, Prop. 3.2] that if the short exact sequence (6.1) splits, the canonical homomorphism $I \to C_p$ has an inverse in the category *Asym* of separable C^*-algebras and homotopy classes of (non-discrete) asymptotic morphisms. On the other hand [136, Thm. 7.2] shows that an isomorphism $B_1 \cong B_2$ in the category *Asym* induces an isomorphism $[[A, B_1]]_{\mathbb{N}} \cong [[A, B_2]]_{\mathbb{N}}$ and hence $[[A, B_1]]_{\mathbb{N}}^{\mathrm{cp}} \cong [[A, B_2]]_{\mathbb{N}}^{\mathrm{cp}}$ if A is nuclear.

If B is nuclear, then so are D, I and C_p and hence I is isomorphic to C_p in the category $Asym^{\mathrm{cp}}$. By using the version of [136, Thm. 7.2] for cpc asymptotic morphisms that was mentioned in Remark 6.2.3, we see that $[[A, C_p]]_{\mathbb{N}}^{\mathrm{cp}} \cong [[A, I]]_{\mathbb{N}}^{\mathrm{cp}}$ if B is nuclear. Therefore if either A or B is nuclear, the following is a short sequence of pointed sets:

$$0 \longrightarrow [[A, I]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, B]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow [[A, D]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow 0. \tag{6.3}$$

In the case of the split extension $0 \to B \otimes \mathbb{K} \to \widetilde{B} \otimes \mathbb{K} \to \mathbb{K} \to 0$, we obtain from (6.3) the following short exact sequence of pointed sets

$$0 \longrightarrow [[A, B \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}} \xrightarrow{\alpha} [[A, \widetilde{B} \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}} \xrightarrow{\beta} [[A, \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}} \longrightarrow 0.$$

Note that $[[A, B \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is an abelian monoid, and by the first part of the proof, $[[A, \widetilde{B} \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ and $[[A, \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ are abelian groups. All monoids are pointed by their respective neutral elements, the map α is a monoid homomorphism and β is a group homomorphism. Exactness implies that the sequence $0 \to [[A, B \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}} \to \ker(\beta) \to 0$ is also exact. By Lemma 6.2.10 we conclude that $[[A, B \otimes \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is an abelian group. □

Proposition 6.2.12. *Let A be a separable C^*-algebra.*
(i) A has property (QH) if and only if A is quasidiagonal and $[[A, \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is a group.
(ii) If A is nuclear, then A has property (QH) if and only if $[[A, A \otimes \mathbb{K}]]_{\mathbb{N}}$ is a group.

Proof. (i) If A has property (QH), then A is quasidiagonal by [144]. Proposition 6.2.11 implies that $[[A, \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is a group. For the other direction, since A is quasidiagonal, there is an injective cpc discrete asymptotic morphism $\{\varphi_n : A \to \mathbb{K}\}_n$. Let (φ_n') be such that $[[\varphi_n']] = -[[\varphi_n]]$ in $[[A, \mathbb{K}]]_{\mathbb{N}}^{\mathrm{cp}}$. Then $(\eta_n) = (\varphi_n \oplus \varphi_n')$ is injective and null-homotopic.

(ii) If A is nuclear, then $[[A, B]]^{\mathrm{cp}} \cong [[A, B]]$ for any separable C^*-algebra B. Suppose that A has property (QH). Proposition 6.2.11 implies that $[[A, A \otimes \mathbb{K}]]_{\mathbb{N}}$ is a group.

For the other direction note that since $[[A, B \otimes \mathbb{K}]]_{\mathbb{N}} \cong [[A \otimes \mathbb{K}, B \otimes \mathbb{K}]]_{\mathbb{N}}$ we may assume that $A \cong A \otimes \mathbb{K}$. Since $[[\mathrm{id}_A]]$ has an additive inverse in $[[A, A]]_{\mathbb{N}}$ it follows that there is an injective cpc discrete asymptotic morphism $\{\varphi_n : A \to A\}_n$ which is null-homotopic. By composing φ_n with a representation of A we find an injective cpc discrete asymptotic morphism $\{\theta_n : A \to L(H)\}_n$ which is null-homotopic. We conclude by applying Proposition 6.2.5. □

6.3 Nuclear homotopy symmetric C^*-algebras

Theorem 6.3.1. *Let A be a separable, nuclear C^*-algebra. Then A has property (QH) if and only if A is homotopy symmetric. In either case, $[[A, B \otimes \mathbb{K}]] \cong E(A, B) \cong KK(A, B)$*

for any separable C^*-algebra B.

Proof. Let A and B be separable C^*-algebras. The sets $[[A, B \otimes \mathbb{K}]]$ and $[[A, B \otimes \mathbb{K}]]_{\mathbb{N}}$ have natural structures of abelian monoids where the addition operation is defined via direct sums. Suppose now that A has property (QH). Then, it follows by Proposition 6.2.11 that $[[A, B \otimes \mathbb{K}]]_{\mathbb{N}}$ becomes an abelian group.

By Lemma 5.6 of [136] for any separable C^*-algebra B there is an exact sequence of pointed sets

$$[[A, SB \otimes \mathbb{K}]]_{\mathbb{N}} \xrightarrow{\alpha} [[A, B \otimes \mathbb{K}]] \xrightarrow{\beta} [[A, B \otimes \mathbb{K}]]_{\mathbb{N}} \xrightarrow{1-\sigma} [[A, B \otimes \mathbb{K}]]_{\mathbb{N}}.$$

Here σ is the shift map $\sigma[[\psi_n]] = [[\psi_{n+1}]]$, β is the natural restriction map and α is defined by stringing together the components of a discrete asymptotic morphism $\{\varphi_n : A \to C_0(0,1) \otimes B \otimes \mathbb{K}\}_n$ to form a continuous asymptotic morphism $\{\Phi_t : A \to B \otimes \mathbb{K}\}_{t \in [0,\infty)}$, where $\Phi_t(a) = \varphi_n(a)(t-n)$ for $t \in [n, n+1]$.

We observe that $(1-\sigma)$ is a morphism of groups and both α and β are monoid homomorphisms. By Lemma 6.2.10, the exact sequence

$$[[A, SB \otimes \mathbb{K}]]_{\mathbb{N}} \to [[A, B \otimes \mathbb{K}]] \to \ker(1-\sigma) \to 0$$

implies that $[[A, B \otimes \mathbb{K}]]$ is a group. In particular, taking $B = A$ we see that A is homotopy symmetric.

Conversely, suppose that A is homotopy symmetric. Then $[[A \otimes \mathbb{K}, A \otimes \mathbb{K}]]$ is a group. The product $[[A, A \otimes \mathbb{K}]]_{\mathbb{N}} \times [[A \otimes \mathbb{K}, A \otimes \mathbb{K}]] \to [[A, A \otimes \mathbb{K}]]_{\mathbb{N}}$, $(x, y) \mapsto y \circ x$ has the property that $(y_1 + y_2) \circ x = y_1 \circ x + y_2 \circ x$. By applying this property with $y_1 = [[\mathrm{id}_{A \otimes \mathbb{K}}]]$ and $y_2 = -y_1$ we obtain that $[[A, A \otimes \mathbb{K}]]_{\mathbb{N}}$ is a group. It follows that A has property (QH) by Proposition 6.2.12(ii).

For the last part of the statement we apply the main result of [45]. □

Remark 6.3.2. Let M be a homotopy associative and homotopy commutative H-space and let X be a topological space with the homotopy type of a CW-complex. By a slight extension of [148, Thm. X.2.4] the homotopy classes of continuous maps $[X, M]$ form a group if and only if $\pi_0(M) = [pt, M]$ is a group. The combination of Proposition 6.2.12 and Theorem 6.3.1 provides a counterpart for asymptotic morphisms of this statement: Let A be a nuclear quasidiagonal C^*-algebra. Then the monoid $[[A, B \otimes \mathbb{K}]]$ is a group for any separable C^*-algebra B if and only if $[[A, \mathbb{K}]] = [[A, C(pt) \otimes \mathbb{K}]]$ is a group. Moreover, we saw that in order to verify the latter condition, one would only need to show that $[[A, \mathbb{K}]]_{\mathbb{N}}$ is a group.

The importance of Theorem 6.3.1 comes from the fact that property (QH) is much easier to verify than the property of being homotopy symmetric. It allows us to vastly extend the class of known homotopy symmetric C^*-algebras.

Theorem 6.3.3. *The class of homotopy symmetric C^*-algebras has the following permanence properties:*

(a) A nuclear C^*-subalgebra of a separable C^*-algebra with property (QH) is homotopy symmetric.

(b) Let $(A_n)_n$ be a sequence of separable C^*-algebras with property (QH). Any separable nuclear C^*-subalgebra of $\prod_n A_n / \bigoplus_n A_n$ is homotopy symmetric.

(c) The class of separable nuclear homotopy symmetric C^*-algebras is closed under inductive limits.

(d) If $0 \to I \to A \to B \to 0$ is a split short exact sequence of separable nuclear C^*-algebras and two of the entries are homotopy symmetric, then so is the third.

(e) The class of homotopy symmetric C^*-algebras is closed under tensor products by separable C^*-algebras and under (asymptotic) homotopy equivalence.

(f) The class of separable nuclear homotopy symmetric C^*-algebras is closed under crossed products by second countable compact groups.

Proof. Since property (QH) passes obviously to C^*-subalgebras, statement (a) follows from Theorem 6.3.1.

Let A be a C^*-subalgebra of $\prod_n A_n / \bigoplus_n A_n$ as in statement (b). Using the Choi-Effros lifting theorem, we find a cpc discrete asymptotic morphism $\{\theta_n : A \to A_n\}_n$ such that $\limsup_n \|\theta_n(a)\| = \|a\|$ for all $a \in A$. Since A is separable, by replacing θ_n by finite direct sums of the form $\theta_n \oplus \theta_{n+1} \oplus \cdots \oplus \theta_N$ and A_n by $A_n \oplus A_{n+1} \oplus \cdots \oplus A_N$ we may assume that $\lim_n \|\theta_n(a)\| = \|a\|$ for all $a \in A$. Here we use the observation that the class of C^*-algebras with property (QH) is closed under finite direct sums. Let $(\Phi_n^{A_i})_n$ be the homotopy of discrete asymptotic morphisms given by Proposition 6.2.5(i) for A_i. Since A is separable, one can find an increasing sequence $m(n)$ of natural numbers such that $\Phi_n := \Phi_{m(n)}^{A_n} \circ \theta_n$ satisfies the condition (i) of Proposition 6.2.5 for A in the sense that (Φ_n) is a homotopy between an injective cpc discrete asymptotic morphism and the null map. We conclude the proof of (b) by applying Theorem 6.3.1.

Statement (c) follows from (b) since any inductive limit $\varinjlim A_n$ embeds as a C^*-subalgebra of $\prod_n A_n / \bigoplus_n A_n$.

For the proof of (d) first note that the cases of nuclear subalgebras and quotients follow from (a), since we assumed that the sequence splits. It remains to be proven that A is homotopy symmetric if I and B are. If B is homotopy symmetric, then $[[B \otimes \mathbb{K}, B \otimes \mathbb{K}]]$ is a group. There is an element $y \in [[B \otimes \mathbb{K}, B \otimes \mathbb{K}]]$ such that $[[\mathrm{id}_{B\otimes\mathbb{K}}]] + y = 0$. Therefore $[[A, B \otimes \mathbb{K}]]$ is a group as well, since $y \circ x$ is an additive inverse of $x \in [[A, B \otimes \mathbb{K}]]$. Likewise $[[A, I \otimes \mathbb{K}]]$ is a group if I is homotopy symmetric. By [45, Prop.3.2] we have a short exact sequence of monoids

$$0 \to [[A, I \otimes \mathbb{K}]] \to [[A, A \otimes \mathbb{K}]] \to [[A, B \otimes \mathbb{K}]] \to 0$$

and Lemma 6.2.10 implies that $[[A, A \otimes \mathbb{K}]]$ is a group. Thus, A is homotopy symmetric.

The statement (e) is an immediate consequence of the definition as noted in [45].

For the proof of (f) let A be a separable C^*-algebra and let G be a second countable compact group that acts on A by automorphisms. Then

$$A \rtimes G \subset (A \otimes C(G)) \rtimes G \cong A \otimes \mathbb{K}(L^2(G))$$

by [66, Cor. 2.9]. We conclude the proof by applying (a). $\square$

The following corollary exhibits a new large class of homotopy symmetric C^*-algebras.

Corollary 6.3.4. *Let A be a separable continuous field of nuclear C^*-algebras over a compact connected metrizable space X. If one of the fibers of A is homotopy symmetric, then A is homotopy symmetric.*

Proof. A has nuclear fibers and hence it is a nuclear C^*-algebra. A embeds in $C(X) \otimes \mathcal{O}_2$ by [19]. Fix $x_0 \in X$ such that $A(x_0)$ is homotopy symmetric. Furthermore, since the embedding is $C(X)$-linear, it follows that A embeds in $E = \{f \in C(X) \otimes \mathcal{O}_2 : f(x_0) \in D\}$, where $D \subset \mathcal{O}_2$ is a C^*-subalgebra isomorphic to $A(x_0)$. Thus, E fits into a short exact sequence

$$0 \longrightarrow C_0(X \setminus \{x_0\}) \otimes \mathcal{O}_2 \longrightarrow E \xrightarrow{ev_{x_0}} D \longrightarrow 0$$

This sequence splits via the $*$-homomorphism $D \to E$ that maps d to the constant function $f(x) = d$ on X. Since both $C_0(X \setminus \{x_0\}) \otimes \mathcal{O}_2$ and D are homotopy symmetric, the statement now follows from Theorem 6.3.3 (d). $\square$

Remark 6.3.5. Statement (c) in Theorem 6.3.3 strengthens the main result of [38] in the case of nuclear C^*-algebras.

6.4 Group C^*-algebras

In this section we prove that any countable torsion free nilpotent group G has property (QH) and hence that the augmentation ideal $I(G)$ is homotopy symmetric. We thank the referee for the suggestion to discuss the connection between the property (QH) for $I(G)$ and the almost flat vector bundles on the classifying space BG.

Theorem 6.4.1. *Let $1 \to N \to G \to H \to 1$ be a central extension of discrete countable amenable groups where N is torsion free. If H has property (QH) then so does G.*

Proof. By [110, Thm. 1.2], (see also [58, Lemma 6.3]), $C^*(G)$ is a nuclear continuous field of C^*-algebras over the spectrum $\widehat{N}$ of $C^*(N)$. Moreover, the fiber over the trivial character ι of N is isomorphic to $C^*(H)$. It follows that $I(G)$ is a nuclear continuous field of C^*-algebras over the spectrum $\widehat{N}$ whose fiber at ι is isomorphic to $I(H)$. Since N is torsion free, its Pontriagin dual is connected. We conclude the proof by applying Cor. 6.3.4. $\square$

Lemma 6.4.2. *Suppose that a countable discrete amenable group G is the union of an increasing sequence of subgroups $(G_i)_i$ each of which has property (QH). Then G has property (QH).*

Proof. Since G is amenable, so is each G_i and the associated group C^*-algebras are nuclear. We may regard $C^*(G_i)$ as a C^*-subalgebra of $C^*(G)$. Then the union of $C^*(G_i)$ is dense in $C^*(G)$ and hence the union of $I(G_i)$ is dense in $I(G)$. The conclusion follows from Thm. 6.3.3 (c). $\square$

Theorem 6.4.3. *If G is a countable torsion free nilpotent group, then $I(G)$ is a homotopy symmetric C^*-algebra.*

Proof. Subgroups of nilpotent groups are nilpotent. Consequently, we may assume by Lemma 6.4.2 that G is finitely generated. Since G is nilpotent it has a finite upper central series $(Z_i)_{i=0}^n$ consisting of subgroups

$$\{1\} = Z_0 \subset Z_1 \subset \cdots \subset Z_{n-1} \subset Z_n = G, \tag{6.4}$$

where Z_1 is the center of G and for $i \geq 1$, Z_{i+1} is the unique subgroup of G such that Z_{i+1}/Z_i is the center of G/Z_i. We argue by induction on the length n of the central series of G. Suppose that property (QH) holds for all finitely generated nilpotent groups with upper central series of length $n-1$. If G is a finitely generated, torsion free and satisfies (6.4), then G/Z_1 is finitely generated, torsion free and nilpotent by [76, Cor. 1.3] and $(Z_{i+1}/Z_1)_{i=0}^{n-1}$ is a central series of length $n-1$ for G/Z_1. Since the upper central series is the shortest central series [118, 5.1.9], we conclude the proof by applying Theorem 6.4.1 to the central extension $1 \to Z_1 \to G \to G/Z_1 \to 1$. $\square$

Remark 6.4.4. (i) The assumption that G is torsion free is essential. Indeed if $s \in G$ is an element of order $n > 1$, then $1 - \frac{1}{n}(1 + s + \cdots + s^{n-1})$ is a nonzero projection contained in $I(G)$.

(ii) If $\pi : \mathbb{H}_3 \to U(n)$ is a representation of the Heisenberg group whose restriction to the center is non-trivial, then there is no continuous path of representations connecting π to a multiple of the trivial representation, [3].

(iii) The K-homology of nilpotent groups such as $\mathbb{H}_{2n+3}$, $n \geq 1$, has nontrivial torsion, [90]. In view of Theorem 6.4.3, there are matricial deformations of $C^*(\mathbb{H}_{2n+3})$ which detect this torsion.

Remark 6.4.5. It is conjectured in [44] that $I(G)$ is homotopy symmetric for all countable amenable torsion free groups G. To prove the conjecture in general, in view of Thorem 6.3.1, it would suffice to find an embedding of $I(G)$ into $\prod_n C\mathbb{K}/\bigoplus_n C\mathbb{K}$. While Theorem 6.4.3 confirms the conjecture for the class of nilpotent groups, we are not yet close to solving the general case. A recent major result of Tikuisis, Winter and White [137] answered positively a conjecture of Rosenberg on the quasidiagonality of amenable groups. This result is equivalent to the statement that $I(G)$ embeds into $\prod_n \mathbb{K}/\bigoplus_n \mathbb{K}$ and can viewed as further evidence for the conjecture in [44].

Connes, Gromov and Moscovici [33] developed and used the concepts of almost flat bundle, almost flat K-theory class and group quasi-representation as tools for proving the Novikov conjecture for large classes of groups. The correspondence between almost flat bundles on a triangularizable compact connected space and the quasi-representations of its fundamental group was revisited in [31].

We have seen in Remark 6.2.8 that if A has property (QH), then A is quasidiagonal. In particular, if a discrete countable group G has property (QH), then $I(G)$ and hence $C^*(G)$ is quasidiagonal. The following theorem is a special case of [44, Thm. 4.5], a result that relies on work of Kasparov [84], Yu [153] and Tu [142].

Theorem 6.4.6 ([44])**.** *Let G be a countable, discrete, torsion free group which is uniformly embeddable in a Hilbert space. Suppose that the classifying space BG can be modeled by a finite simplicial complex and that the full group C^*-algebra $C^*(G)$ is quasidiagonal. Then all the elements of $K^0(BG)$ are almost flat.*

If G is a countable, discrete, torsion free amenable group, then G is uniformly embeddable in a Hilbert space and its C^*-algebra is quasidiagonal by [137]. Therefore:

Corollary 6.4.7. *If a countable, discrete, torsion free amenable group G admits a finite classifying space BG, then all the elements of $K^0(BG)$ are almost flat.*

In particular, this corollary applies to all countable torsion free finitely generated nilpotent groups as they are known to admit finite classifying spaces. The C^*-algebras of such groups G are known to be residually finite dimensional, hence quasidiagonal, as G is amenable and residually finite.

7 Crossed module actions on continuous trace C^*-algebras

We lift an action of a torus $\mathbb{T}^n$ on the spectrum of a continuous trace algebra to an action of a certain crossed module of Lie groups that is an extension of $\mathbb{R}^n$. We compute equivariant Brauer and Picard groups for this crossed module and describe the obstruction to the existence of an action of $\mathbb{R}^n$ in our framework.

7.1 Introduction

Let P be a principal $\mathbb{T}^n$-bundle over X and let A be a continuous trace C^*-algebra with spectrum P. The action of $\mathbb{T}^n$ does not always lift to a $\mathbb{T}^n$-action on A: this may fail already for $n = 1$. Lifting the action to an action of $\mathbb{R}^n$ on A works much more often: for this, we only need to know that A restricts to trivial bundles over the orbits of the action (see [36]). If an $\mathbb{R}^n$-action on A exists and if $\hat{A} = A \rtimes \mathbb{R}^n$ is again a continuous trace C^*-algebra with spectrum $\hat{P}$, then $\hat{A}$ is considered T-dual to A, and the Connes–Thom isomorphism $\mathbb{K}_*(A) \cong \mathbb{K}_{*+n}(\hat{A})$ is the T-duality isomorphism between the twisted K-theory groups of P and $\hat{P}$ given by A and $\hat{A}$, respectively, see [97]. The C^*-algebra $\hat{A}$ may fail to have continuous trace, for instance, by being a principal bundle of noncommutative tori. In this case, one may still consider the C^*-algebra $\hat{A}$ to be a T-dual for A.

If there is no action of $\mathbb{R}^n$ on A, then [22] suggests non-associative algebras that may play the role of the T-dual. It seems hard to do anything with such non-associative algebras, however. Another situation where non-associative algebras appear naturally are Fell bundles over crossed modules or, equivalently, actions of crossed modules on C^*-algebras (see [28, 30]). We relate these two appearances of non-associativity: the action of $\mathbb{T}^n$ always lifts to an action of a certain crossed module on A, and this action, viewed as a non-associative Fell bundle over $\mathbb{R}^n$, gives the non-associative algebras studied in [22]. Crossed module actions may give a good framework to understand the non-associative algebras proposed in [22].

Even in the applications to T-duality, it is useful to allow non-principal $\mathbb{T}^n$-bundles. We treat this more general case right away, that is, we allow P to be an arbitrary second countable, locally compact $\mathbb{T}^n$-space. Any $\mathbb{T}^n$-action lifts to an action of our crossed module.

A crossed module of topological groups $\mathcal{H} = (H^1, H^2, \partial)$ is given by topological groups H^1 and H^2 with continuous group homomorphisms $\partial\colon H^2 \to H^1$ and $c\colon H^1 \to \mathrm{Aut}(H^2)$ with two conditions that mimic the properties of a normal subgroup and the conjugation action on the subgroup. We shall mostly use the following crossed module:

$$\begin{aligned} H^1 &= \mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n, & (t_1,\eta_1)\cdot(t_2,\eta_2) &= (t_1+t_2, \eta_1+\eta_2+t_1\wedge t_2),\\ H^2 &= \Lambda^2\mathbb{R}^n \oplus \Lambda^3\mathbb{R}^n, & (\theta_1,\xi_1)\cdot(\theta_2,\xi_2) &= (\theta_1+\theta_2, \xi_1+\xi_2),\\ \partial\colon H^2 &\to H^1, & \partial(\theta,\xi) &= (0,\theta),\\ c\colon H^1 &\to \mathrm{Aut}(H^2), & c_{(t,\eta)}(\theta,\xi) &= (\theta, \xi + t\wedge\theta), \end{aligned} \tag{7.1}$$

where $t, t_1, t_2 \in \mathbb{R}^n$, $\eta, \eta_1, \eta_2, \theta, \theta_1, \theta_2 \in \Lambda^2\mathbb{R}^n$ and $\xi, \xi_1, \xi_2 \in \Lambda^3\mathbb{R}^n$. Here $\Lambda^k\mathbb{R}^n$ denotes the kth exterior power of $\mathbb{R}^n$. An action of this crossed module on a C^*-algebra A consists of two continuous group homomorphisms $\alpha\colon H^1 \to \mathrm{Aut}(A)$ and $u\colon H^2 \to U(M(A))$ with $\alpha_{\partial(h)} = \mathrm{Ad}_{u(h)}$ for all $h \in H^2$ and $\alpha_h(u_k) = u_{c_h(k)}$ for all $h \in H^1$, $k \in H^2$; in addition, we want α to lift the given action of $\mathbb{R}^n$ on the spectrum of A.

A representation (α, u) as above is determined uniquely by $\alpha_t = \alpha_{(t,0)}$ for $t \in \mathbb{R}^n$ and $u_{t_1\wedge t_2} = u_{(t_1\wedge t_2,0)}$ for $t_1, t_2 \in \mathbb{R}^n$ because $\alpha_{t_1}\alpha_{t_2}\alpha_{t_1+t_2}^{-1} = \alpha_{(0,t_1\wedge t_2)}$ and $\alpha_{t_1}(u_{t_2\wedge t_3}) = u_{(0,t_1\wedge t_2\wedge t_3)}$. The unitaries $u_{t_1\wedge t_2}$ are such that $\mathrm{Ad}_{u_{t_1\wedge t_2}} = \alpha_{(0,t_1\wedge t_2)}$. So $t \mapsto \alpha_t$ is an action of $\mathbb{R}^n$ up to inner automorphisms given by $u_{t_1\wedge t_2}$. These unitaries are, however, not themselves α_t-invariant. We only know that the unitaries $\alpha_{t_1}(u_{t_2\wedge t_3}) = u_{(0,t_1\wedge t_2\wedge t_3)}$ are central.

If we divide out the Λ^3-part in H^2, then we get a crossed module that is equivalent to $\mathbb{R}^n$; its actions are Green twisted actions of $\mathbb{R}^n$, so they can be turned into ordinary actions of $\mathbb{R}^n$ on a C^*-stabilisation. The assumptions above imply that $\Lambda^3\mathbb{R}^n$ acts through a map to central, $\mathbb{R}^n$-invariant unitaries in A, that is, by a map to $C(P/\mathbb{R}^n, \mathbb{T})$. A homomorphism $\Lambda^3\mathbb{R}^n \to C(P/\mathbb{R}^n, \mathbb{T})$ lifts uniquely to $\Lambda^3\mathbb{R}^n \to C(P/\mathbb{R}^n, \mathbb{R})$, and such a homomorphism appears as the obstruction to finding an action of $\mathbb{R}^n$ on A that lifts the given $\mathbb{T}^n$-action on P. Actions of the crossed module $\mathcal{H}$ may contain such a lifting obstruction, so that there is no longer any obstruction to lifting the action to one of $\mathcal{H}$.

Besides proving the existence of liftings, we also classify them up to equivalence; that is, we compute the equivariant Brauer group with respect to the crossed module $\mathcal{H}$: There is an exact sequence of Abelian groups

$$H^2(P,\mathbb{Z}) \to C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n) \to \mathrm{Br}_{\mathcal{H}}(P) \twoheadrightarrow \mathrm{Br}(P).$$

The surjection $\mathrm{Br}_{\mathcal{H}}(P) \twoheadrightarrow \mathrm{Br}(P)$ says that any continuous trace C^*-algebra over P carries an action of $\mathcal{H}$ lifting the given action of $\mathbb{T}^n$ on the spectrum. The description of the kernel is the same as for the $\mathbb{R}^n$-equivariant Brauer group, so our result says that whenever we may lift the action on P to an $\mathbb{R}^n$-action on A, then there is a bijection between actions of $\mathcal{H}$

and $\mathbb{R}^n$ on A.

We also compute the analogue of the equivariant Picard group for our crossed module actions, and get the same result as in the case of $\mathbb{R}^n$-actions. Thus the only effect of replacing $\mathbb{R}^n$ by the crossed module $\mathcal{H}$ is to remove the obstruction to the existence of actions.

Our proof uses a smaller weak 2-group $\mathcal{G}$ that is equivalent to $\mathcal{H}$. It consists of the Abelian groups $G^1 = \operatorname{coker}(\partial_{\mathcal{H}}) \cong \mathbb{R}^n$ and $G^2 = \ker(\partial_{\mathcal{H}}) \cong \Lambda^3\mathbb{R}^n$, which are linked by the non-trivial associator

$$a(t_1, t_2, t_3) = -t_1 \wedge t_2 \wedge t_3 \qquad \text{for } t_1, t_2, t_3 \in \mathbb{R}^n.$$

We describe morphisms that give an equivalence between $\mathcal{G}$ and the weak 2-group associated to $\mathcal{H}$. The crossed module actions above are strict, but there is a more flexible notion of weak action that makes sense for weak 2-groups and 2-groupoids as well, see [29]. The Packer–Raeburn Stabilisation Trick extends to crossed modules and shows that any weak action of a crossed module is equivalent to a strict action. Since $\mathcal{G}$ and $\mathcal{H}$ are equivalent, they have equivalent weak actions on C^*-algebras. Thus we get the desired strict action of $\mathcal{H}$ once we construct a weak action of the transformation bigroupoid $\mathcal{G} \ltimes P$ associated to P.

The weak actions we have in mind are equivalent to saturated Fell bundles in the case of a group action. For actions of a 2-group such as $\mathcal{G}$, they are non-associative Fell bundles over $G^1 = \mathbb{R}^n$ where the associator is given by unitaries coming from the action of $G^2 = \Lambda^3\mathbb{R}^n$. Allowing weak actions greatly simplifies the study of equivariant Brauer groups, already in the group case. We reprove the obstruction theory for $\mathbb{R}^n$-actions on continuous trace C^*-algebras along the way. The bigroupoid version of the result is only notationally more difficult.

The crossed module $\mathcal{H}$ described above can be made slightly smaller: we may divide out the lattice $\Lambda^3\mathbb{Z}^n$ inside $\Lambda^3\mathbb{R}^n$, resulting in a compact group. This is so because the lifting obstructions that appear for $\mathbb{T}^n$-actions on continuous trace algebras always vanish on the lattice $\Lambda^3\mathbb{Z}^n$, and hence so do all the actions of $\mathcal{H}$ that appear. This feature, however, is special to actions of $\mathbb{R}^n$ that factor through the standard torus $\mathbb{R}^n/\mathbb{Z}^n$. The existence result for actions of $\mathcal{H}$ still works for actions of $\mathbb{R}^n$ if the stabiliser lattices are allowed to vary continuously for different orbits.

To compute the lifting obstruction of a given continuous trace C^*-algebra, it suffices to consider a single free $\mathbb{T}^n$-orbit, that is, the case of the standard translation action of $\mathbb{T}^n$ on itself. Since $\mathbb{R}^n$ acts transitively on $\mathbb{T}^n$, the transformation crossed module $\mathcal{H} \ltimes \mathbb{T}^n$ is equivalent in a suitable sense to the "stabiliser" of a point in $\mathbb{T}^n$; this gives the subcrossed module $\tilde{\mathcal{H}}$ of $\mathcal{H}$ with $\tilde{H}^1 = \mathbb{Z}^n \times \Lambda^2\mathbb{R}^n$ and $\tilde{H}^2 = H^2$. We do not develop the full theory of induction for crossed modules because the relevant special case is easy to do by hand. It turns out that actions of $\mathcal{H}$ on continuous trace C^*-algebras over $\mathbb{T}^n$ are equivalent to actions of $\tilde{\mathcal{H}}$ on continuous trace C^*-algebras over the point. Since our whole theory is up to equivalence, this is the same as weak actions of $\tilde{\mathcal{H}}$ on the complex numbers $\mathbb{C}$. Replacing $\tilde{\mathcal{H}}$ by the corresponding sub-2-group $\tilde{\mathcal{G}} \subseteq \mathcal{G}$, it is straightforward to classify these. This also gives the equivariant Brauer group for a single $\mathbb{T}^n$-orbit, and then allows to identify the lifting obstruction, up to a sign maybe, with the family of Dixmier–Douady invariants of the restrictions of A to the orbits of the $\mathbb{R}^n$-action.

7.2 A crossed module extension of $\mathbb{R}^n$

We construct a crossed module extension $\mathcal{H}$ of $\mathbb{R}^n$ that circumvents the obstruction to lifting $\mathbb{R}^n$-actions from spaces to continuous trace C^*-algebras described in [36]. We also describe a smaller weak topological 2-group $\mathcal{G}$ equivalent to $\mathcal{H}$.

Definition 7.2.1. A *crossed module* of topological groups $\mathcal{H} = (H^1, H^2, \partial)$ is given by topological groups H^1 and H^2 with continuous group homomorphisms $\partial\colon H^2 \to H^1$ and $c\colon H^1 \to \operatorname{Aut}(H^2)$, such that

$$\partial(c_{h_1}(h_2)) = h_1\, \partial(h_2)\, h_1^{-1}, \qquad c_{\partial(h_2)}(h_2') = h_2\, h_2'\, h_2^{-1}$$

for all $h_1 \in H^1$, $h_2, h_2' \in H^2$; continuity of c means that the map $H^1 \times H^2 \to H^2$, $(h_1, h_2) \mapsto c_{h_1}(h_2)$, is continuous.

A weak 2-group is a bicategory with only one object and with invertible arrows and 2-arrows. A 2-arrow or 2-cell in a bicategory is an “arrow” between two arrows. We call them “bigons” because they should be drawn like this:

$$\begin{tikzcd} y & x; \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow a"] \end{tikzcd}$$

here x and y are objects, f and g are arrows $x \to y$, and a is a bigon from f to g. The standard reference for bicategories is [15]; we mainly follow [29], which is concerned with their C^*-algebraic applications.

Crossed modules model strict 2-groups, that is, 2-groups with trivial unitors and associator. Let $\mathcal{H} = (H^1, H^2, \partial)$ be a crossed module. The associated strict 2-group $\mathcal{C}_{\mathcal{H}}$ has a single object and H^1 as its space of arrows. The space of bigons is $H^2 \times H^1$ with source map

$(h_2, h_1) \mapsto h_1$ and target map $(h_2, h_1) \mapsto \partial(h_2)h_1$. We denote this by $h_2 \colon h_1 \Rightarrow \partial(h_2)h_1$ or

$$\begin{tikzcd} * & * \arrow[l, bend right, "h_1"'] \arrow[l, bend left, "\partial(h_2)h_1"] \arrow[l, phantom, "\Downarrow h_2"] \end{tikzcd}.$$

The composition of arrows is the group multiplication in H^1. The vertical composition of bigons is the multiplication in H^2, and the horizontal composite of $h_2 \colon h_1 \Rightarrow \partial(h_2)h_1$ and $k_2 \colon k_1 \Rightarrow \partial(k_2)k_1$ is $h_2 \bullet k_2 = h_2 c_{h_1}(k_2) \colon h_1 k_1 \Rightarrow \partial(h_2)h_1\partial(k_2)k_1 = \partial(h_2 c_{h_1}(k_2))h_1 k_1$.

We shall mostly use the crossed module $\mathcal{H}$ defined in (7.1). It is routine to check that $\mathcal{H}$ is a crossed module of Lie groups.

Since the arrow (t, η) in the Lie 2-group $\mathcal{C}_{\mathcal{H}}$ is equivalent to $(t, 0)$ by a bigon, we should be able to shrink $\mathcal{C}_{\mathcal{H}}$, replacing the space of arrows by $\mathbb{R}^n$. This should reduce the space of bigons to $\mathbb{R}^n \times \Lambda^3\mathbb{R}^n$ because bigons $(t_1, 0) \Rightarrow (t_2, 0)$ in $\mathcal{C}_{\mathcal{H}}$ are of the form $(0, \xi)$. The result of this shrinking is a Lie 2-group $\mathcal{G}$ equivalent to $\mathcal{C}_{\mathcal{H}}$. This is no longer a strict Lie 2-group, however, so it does not come directly from a crossed module. The starting point is the smooth map Φ sending an arrow (t, η) to the bigon

$$\begin{tikzcd} * & * \arrow[l, bend right, "(t,\eta)"'] \arrow[l, bend left, "(t,0)"] \arrow[l, phantom, "\Downarrow (-\eta,0)"] \end{tikzcd}.$$

This generates a morphism $F = \mathrm{Ad}_\Phi \colon \mathcal{C}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{H}}$ of Lie 2-groups that is equivalent to the identity functor, which we now describe (morphisms between weak 2-categories are described in [29, Def. 4.1]). The morphism maps $(t, \eta) \in H^1$ to the range $(t, 0) \in H^1$ of the bigon $\Phi(t, \eta)$. It maps a bigon $(\theta, \xi) \colon (t, \eta) \Rightarrow (t, \eta + \theta)$ to the vertical composite

$$(t, 0) \xLeftarrow{\Phi(t,\eta)} (t, \eta) \xRightarrow{(\theta,\xi)} (t, \eta + \theta) \xRightarrow{\Phi(t,\eta+\theta)} (t, 0),$$

which gives $(0, \xi) \colon (t, 0) \Rightarrow (t, 0)$. Since $\Phi(t, 0) = (0, 0)$, our morphism is strictly unital, so the bigon $u_* \colon 1_* \Rightarrow F(1_*)$ in the definition of a morphism is trivial. A morphism F also needs natural bigons

$$\omega_F((t_1, \eta_1), (t_2, \eta_2)) \colon F(t_1, \eta_1) \cdot F(t_2, \eta_2) \Rightarrow F((t_1, \eta_1) \cdot (t_2, \eta_2))$$

describing its compatibility with the multiplication. We get

$$\omega_F((t_1, \eta_1), (t_2, \eta_2)) \colon (t_1, 0) \cdot (t_2, 0) = (t_1 + t_2, t_1 \wedge t_2) \Rightarrow (t_1 + t_2, 0)$$

by vertically composing the following 2-arrows:

$$\begin{aligned}(t_1,0)\cdot(t_2,0) &\xRightarrow{\Phi(t_1,\eta_1)^{-1}\bullet\Phi(t_2,\eta_2)^{-1}} (t_1,\eta_1)\cdot(t_2,\eta_2)\\ &\qquad = (t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2) \xRightarrow{\Phi(t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2)} (t_1+t_2,0);\end{aligned}$$

this gives $c_{(t_1,0)}(\eta_2,0)+(\eta_1,0)-(\eta_1+\eta_2+t_1\wedge t_2,0)=(-t_1\wedge t_2,t_1\wedge\eta_2)$, so

$$\omega_F((t_1,\eta_1),(t_2,\eta_2))=(-t_1\wedge t_2,t_1\wedge\eta_2).$$

By construction, Φ is a transformation from the identity functor to the functor F. Since all bigons in $\mathcal{C}_{\mathcal{H}}$ are invertible, this is even an equivalence. We are going to describe the range of F, which is a weak Lie 2-group $\mathcal{G}$. Its arrows and bigons are

$$\begin{aligned}G^1&=\mathbb{R}^n=\{(t,0)\mid t\in\mathbb{R}^n\}\subseteq H^1,\\ G^2&=\mathbb{R}^n\times\Lambda^3\mathbb{R}^n=\{((t,0),(0,\xi))\mid t\in\mathbb{R}^n,\xi\in\Lambda^3\mathbb{R}^n\}\subseteq H^1\times H^2;\end{aligned}$$

so $(t,\xi)\in\mathbb{R}^n\times\Lambda^3\mathbb{R}^n$ corresponds to the bigon $(0,\xi)\colon(t,0)\Rightarrow\partial(0,\xi)(t,0)=(t,0)$ and thus has range and source t. The vertical composition of bigons adds the ξ-components as in $\mathcal{H}$. The composition of arrows gives

$$(t_1,0)\cdot_{\mathcal{G}}(t_2,0)=F((t_1,0)\cdot(t_2,0))=F(t_1+t_2,t_1\wedge t_2)=(t_1+t_2,0),$$

so it is simply the addition in $\mathbb{R}^n$. The horizontal composition of bigons in $\mathcal{G}$ is also defined by applying F to the horizontal composition in $\mathcal{C}_{\mathcal{H}}$. Since $c_{(t,0)}(0,\xi)=(0,\xi)$ for all $t\in\mathbb{R}^n$, $\xi\in\Lambda^3\mathbb{R}^n$, the horizontal composition in $\mathcal{G}$ also simply adds the ξ-components. The unit arrow in $\mathcal{G}$ is $(0,0)$ and the left and right unitors are the identity bigons $(0,0)$ because the morphism F is strictly unital. The associator $a(t_1,t_2,t_3)$ for $t_1,t_2,t_3\in\mathbb{R}^n$ is defined so that the following diagram of 2-arrows commutes:

$$\begin{tikzcd}
F((t_1,0)\cdot F((t_2,0)\cdot(t_3,0))) & F(F((t_1,0)\cdot(t_2,0))\cdot(t_3,0)) \arrow[l, Rightarrow, "a(t_1,t_2,t_3)"'] \\
(t_1,0)\cdot F((t_2,0)\cdot(t_3,0)) \arrow[u, Rightarrow, "\Phi(t_1+t_2+t_3,t_1\wedge(t_2+t_3))"'] & F((t_1,0)\cdot(t_2,0))\cdot(t_3,0) \arrow[u, Rightarrow, "\Phi(t_1+t_2+t_3,(t_1+t_2)\wedge t_3)"] \\
(t_1,0)\cdot((t_2,0)\cdot(t_3,0)) \arrow[u, Rightarrow, "1_{(t_1,0)}\bullet\Phi(t_2+t_3,t_2\wedge t_3)"'] \arrow[r, equal] & ((t_1,0)\cdot(t_2,0))\cdot(t_3,0) \arrow[u, Rightarrow, "\Phi(t_1+t_2,t_1\wedge t_2)\bullet 1_{(t_3,0)}"]
\end{tikzcd}$$

Since this involves $c_{(t_1,0)}(-t_2 \wedge t_3, 0) = (-t_2 \wedge t_3, -t_1 \wedge t_2 \wedge t_3)$, we get

$$a(t_1,t_2,t_3) = (-t_1 \wedge (t_2+t_3), 0) - (-(t_1+t_2)\wedge t_3, 0) \\ + (-t_2 \wedge t_3, -t_1 \wedge t_2 \wedge t_3) - (-t_1 \wedge t_2, 0) = (0, -t_1 \wedge t_2 \wedge t_3).$$

These computations lead us to the following definition:

Definition 7.2.2. Let $\mathcal{G}(*,*)$ be the Lie groupoid given by the constant bundle of Abelian Lie groups $\Lambda^3\mathbb{R}^n$ over $\mathbb{R}^n$. Let $\mathcal{G}$ be the weak topological 2-group with one object $*$ and $\mathcal{G}(*,*)$ as its groupoid of arrows and bigons. The composition functor $\cdot\colon \mathcal{G}(*,*)\times\mathcal{G}(*,*)\to\mathcal{G}(*,*)$ is addition in both components. The unitors l and r (called identity transformations in [15]) are trivial. The associator (called associativity isomorphism in [15]) is

$$\alpha\colon \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\times\Lambda^3\mathbb{R}^n, \quad (t_1,t_2,t_3)\mapsto(t_1+t_2+t_3, -t_1\wedge t_2\wedge t_3).$$

It is routine to check that $\mathcal{G}$ is a weak 2-category (see [29, Def. 2.7]) or a bicategory in the notation of [15, (1.1)]. All arrows and bigons are invertible and $\mathcal{G}$ has only one object, so it is a weak Lie 2-group.

The computations above suggest that $\mathcal{G}$ should be equivalent to the strict 2-group $\mathcal{C}_\mathcal{H}$ associated to the crossed module $\mathcal{H}$. More precisely, we are going to construct morphisms $\iota\colon\mathcal{G}\to\mathcal{C}_\mathcal{H}$ and $\pi\colon\mathcal{C}_\mathcal{H}\to\mathcal{G}$ such that $\pi\circ\iota$ is the identity morphism on $\mathcal{G}$ and $\iota\circ\pi = \mathrm{Ad}_\Phi$ is equivalent to the identity morphism on $\mathcal{C}_\mathcal{H}$ by the transformation Φ. The morphism ι maps $t\mapsto(t,0)$ and $\xi\colon t\Rightarrow t$ to $(0,\xi)\colon(t,0)\Rightarrow(t,0)$. This is strictly unital, so the bigon $\iota(1_*)\Rightarrow 1_*$ is $(0,0)$. The compatibility with multiplication is given by

$$\omega_\iota(t_1,t_2) = (-t_1\wedge t_2, 0)\colon (t_1+t_2, t_1\wedge t_2) = \iota(t_1)\cdot\iota(t_2) \\ \Rightarrow \iota(t_1+t_2) = (t_1+t_2, 0).$$

Routine computations show that ι is a morphism; in particular, the cocycle condition [29, (4.2)] for ω_ι becomes

$$c_{(t_1,0)}(\omega_\iota(t_2,t_3)) - \omega_\iota(t_1+t_2,t_3) + \omega_\iota(t_1,t_2+t_3) - \omega_\iota(t_1,t_2) \\ - (0, -t_1\wedge t_2\wedge t_3) = 0,$$

which holds because of our choice of the associator in $\mathcal{G}$.

The projection $\pi\colon\mathcal{C}_\mathcal{H}\to\mathcal{G}$ is defined so that $\iota\circ\pi = \mathrm{Ad}_\Phi$. Since ι is faithful on arrows and bigons, this determines π uniquely, and it implies that π is a morphism because Ad_Φ is one. Our Ansatz dictates $\pi(t,\eta) = t$ and

$$\pi((\theta,\xi)\colon (t,\eta) \Rightarrow (t,\eta+\theta)) = (\xi\colon t \Rightarrow t),$$

so that $\iota \circ \pi$ and Ad_Φ involve the same maps on arrows and bigons. The morphism π must be strictly unital because ι and Ad_Φ are so. The canonical bigon

$$\omega_{\iota\pi}((t_1,\eta_1),(t_2,\eta_2))\colon \iota\pi(t_1,\eta_1)\cdot\iota\pi(t_2,\eta_2) \Rightarrow \iota\pi((t_1,\eta_1)\cdot(t_2,\eta_2))$$

is the vertical composite of

$$\begin{aligned} &\omega_\iota(\pi(t_1,\eta_1),\pi(t_2,\eta_2))\colon \iota\pi(t_1,\eta_1)\cdot\iota\pi(t_2,\eta_2) \Rightarrow \iota(\pi(t_1,\eta_1)\cdot\pi(t_2,\eta_2)),\\ &\iota(\omega_\pi((t_1,\eta_1),(t_2,\eta_2)))\colon \iota(\pi(t_1,\eta_1)\cdot\pi(t_2,\eta_2)) \Rightarrow \iota(\pi((t_1,\eta_1)\cdot(t_2,\eta_2))).\end{aligned}$$

So we must put $\omega_\pi((t_1,\eta_1),(t_2,\eta_2)) = t_1\wedge\eta_2\colon t_1+t_2 \Rightarrow t_1+t_2$ to get $\omega_{\iota\pi} = \omega_{\mathrm{Ad}_\Phi}$. This finishes the construction of π.

Since $\iota\pi = \mathrm{Ad}_\Phi$, Φ is a transformation from the identity on $\mathcal{C}_\mathcal{H}$ to $\iota\pi$. The composite $\pi\iota\colon \mathcal{G} \to \mathcal{G}$ is the identity on objects and arrows, strictly unital, and involves the identical transformation $\omega_{\pi\iota}$, so it is equal to the identity functor. Thus ι and π are equivalences of weak 2-groups inverse to each other. Both are given by smooth maps, so we have an equivalence of Lie 2-groups.

Remark 7.2.3. In the arguments above, we may replace $\Lambda^3\mathbb{R}^n$ everywhere by $\Lambda^3\mathbb{R}^n/\Gamma$ for any closed subgroup Γ. We shall be interested, in particular, in the case where we use $\Lambda^3\mathbb{R}^n/\Lambda^3\mathbb{Z}^n$ because the latter group is compact and because the actions of $\mathcal{G}$ or $\mathcal{H}$ that we need factor through this quotient in the case of $\mathbb{T}^n$-bundles.

7.3 Equivariant Brauer groups for bigroupoids

The equivariant Brauer group $\mathrm{Br}_G(P)$ of a transformation group $G \ltimes P$ is defined in [36]. It classifies continuous trace C^*-algebras with spectrum P with a G-action that lifts the given action on P. This definition is extended from transformation groups to locally compact groupoids in [87]. Here we extend it further to locally compact bigroupoids. "Bigroupoids" are called "weak 2-groupoids" in [29]. The bigroupoids we need combine the groupoid $\mathcal{G}$ defined in Section 7.2 with an action $\alpha\colon \mathbb{R}^n \to \mathrm{Homeo}(P)$ of $\mathbb{R}^n$ by homeomorphisms on a space P.

Explicitly, we consider the following locally compact bigroupoid $\mathcal{C}$. It has object space $\mathcal{C}^0 = P$, arrow space $\mathcal{C}^1 = \mathbb{R}^n \times P$, and space of bigons $\mathcal{C}^2 = \Lambda^3\mathbb{R}^n \times \mathbb{R}^n \times P$. Here an arrow $(t,p) \in \mathbb{R}^n \times P$ has source p and range $\alpha_t(p)$, and the multiplication is the usual one:

$(t_1, \alpha_{t_2}(p_2))\cdot(t_2, p_2) = (t_1+t_2, p_2)$; a bigon $(\xi, t, p) \in \Lambda^3\mathbb{R}^n\times\mathbb{R}^n\times P$ has source and range (t, p); the vertical composition adds the ξ-components: $(\xi_1, t, p) \cdot (\xi_2, t, p) = (\xi_1 + \xi_2, t, p)$; and the horizontal composition is

$$(\xi_1, t_1, \alpha_{t_2}(p_2)) \bullet (\xi_2, t_2, p_2) = (\xi_1 + \xi_2, t_1 + t_2, p_2).$$

The unit arrow on an object p is $(0, p)$, the unit bigon on an arrow (t, p) is $(0, t, p)$. Units are strict, that is, the left and right unitors are trivial. The associator is

$$a((t_1, p_1), (t_2, p_2), (t_3, p_3)) = (-t_1 \wedge t_2 \wedge t_3, t_1 + t_2 + t_3, p_3)$$

for a triple of composable arrows; that is, for $p_i \in P$, $t_i \in \mathbb{R}^n$ for $i = 1, 2, 3$ with $p_{i-1} = \alpha_{t_i}(p_i)$ for $i = 2, 3$. It is routine to check that this is a bigroupoid. All the spaces are smooth manifolds, hence locally compact, all the operations are smooth maps, hence continuous, and range and source maps are surjective submersions, hence open. Thus $\mathcal{C} = \mathcal{G} \ltimes P$ is a Lie bigroupoid and a locally compact bigroupoid.

Actions of locally compact bigroupoids on C^*-algebras are defined in [29]. We shall use [29, Def. 4.1] for the correspondence bicategory as target bicategory ("bicategories" are called "weak 2-categories" in [29]), but with some changes. First, we require functors to be strictly unital, that is, the bigons u_x in [29, Def. 4.1] are identities; this is no restriction of generality because any functor is equivalent to one with this property, as remarked in [29]. Secondly, since $\mathcal{C}$ has invertible arrows, all correspondences appearing in an action of $\mathcal{C}$ are equivalences, that is, imprimitivity bimodules. Third, we need continuous actions; continuity is explained at the end of Section 4.1 in [29]. Fourth, we work with the opposite bicategory of imprimitivity bimodules, that is, an A, B-imprimitivity bimodule is viewed as an arrow from B to A; otherwise the multiplication maps below would go from $E_g \otimes E_h$ to E_{hg} instead of E_{gh}.

Now we define a *continuous action of a bigroupoid* $\mathcal{C}$ *by correspondences* or, equivalently, by imprimitivity bimodules, with the modifications mentioned above. Such an action requires the following data:

- a $C_0(\mathcal{C}^0)$-C^*-algebra A; we denote its fibres by A_x for $x \in \mathcal{C}^0$;
- a $C_0(\mathcal{C}^1)$-linear imprimitivity bimodule E between the pull-backs $r^*(A)$ and $s^*(A)$ of A along the range and source maps; thus E is a bundle over $\mathcal{C}^1$ where the fibre at $g \in \mathcal{C}^1$ is an $A_{r(g)}, A_{s(g)}$-imprimitivity bimodule E_g;
- isomorphisms $\omega_{g,h}\colon E_g \otimes_{A_{s(g)}} E_h \to E_{gh}$ of imprimitivity bimodules for all $g, h \in \mathcal{C}^1$ with $s(g) = r(h)$, which are continuous in the sense that pointwise application of $\omega_{g,h}$

gives an isomorphism of imprimitivity bimodules

$$\omega\colon \mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E) \to \mu^*(E),$$

where $\mathrm{pr}_1, \mathrm{pr}_2, \mu$ are the continuous maps that map a pair (g, h) of composable arrows to g, h and gh, respectively; so $s\,\mathrm{pr}_1 = r\,\mathrm{pr}_2$ maps (g, h) to $s(g) = r(h)$;

- isomorphisms of imprimitivity bimodules $U_b\colon E_{s_2(b)} \to E_{r_2(b)}$ for all bigons $b \in \mathcal{C}^2$, which are continuous in the sense that they give an isomorphism of imprimitivity bimodules $U\colon s_2^*(E) \to r_2^*(E)$; here $r_2, s_2\colon \mathcal{C}^2 \to \mathcal{C}^1$ map a bigon to its range and source arrow;

this data is subject to the following algebraic conditions:

(A1) $E_{1_x} = A_x$ for all $x \in \mathcal{C}^0$, and the restriction of E to units is A;

(A2) $\omega_{g,1}\colon E_g \otimes_{A_{s(g)}} A_{s(g)} \to E_g$ and $\omega_{1,h}\colon A_{r(h)} \otimes_{A_{r(h)}} E_h \to E_h$ are the canonical isomorphisms for all $g, h \in \mathcal{C}^1$;

(A3) U is multiplicative for vertical products: $U_{b_1} \circ U_{b_2} = U_{b_1 \cdot b_2}$ for vertically composable bigons $b_1, b_2 \in \mathcal{C}^2$;

(A4) if $b_1\colon f_1 \Rightarrow g_1$ and $b_2\colon f_2 \Rightarrow g_2$ are horizontally composable bigons, that is, $s(f_1) = s(g_1) = r(f_2) = r(g_2)$, then the following diagram commutes:

$$\begin{tikzcd}
E_{f_1} \otimes_{A_{s(f_1)}} E_{f_2} \arrow[r, "\omega_{f_1,f_2}"] \arrow[d, "U_{b_1} \otimes U_{b_2}"'] & E_{f_1 f_2} \arrow[d, "U_{b_1 \bullet b_2}"] \\
E_{g_1} \otimes_{A_{s(g_1)}} E_{g_2} \arrow[r, "\omega_{g_1,g_2}"] & E_{g_1 g_2}
\end{tikzcd}$$

(A5) if g_1, g_2, g_3 are composable arrows in $\mathcal{C}^1$, then the following diagram commutes:

$$\begin{tikzcd}
(E_{g_1} \otimes E_{g_2}) \otimes E_{g_3} \arrow[r, leftrightarrow] \arrow[d, "\omega_{g_1,g_2} \otimes 1"] & E_{g_1} \otimes (E_{g_2} \otimes E_{g_3}) \arrow[d, "1 \otimes \omega_{g_2,g_3}"] \\
E_{g_1 g_2} \otimes E_{g_3} \arrow[d, "\omega_{g_1 g_2, g_3}"] & E_{g_1} \otimes E_{g_2 g_3} \arrow[d, "\omega_{g_1, g_2 g_3}"] \\
E_{(g_1 g_2) g_3} \arrow[r, "U_{a(g_1,g_2,g_3)}"] & E_{g_1 (g_2 g_3)}
\end{tikzcd}$$

here $a(g_1, g_2, g_3)\colon (g_1 g_2) g_3 \to g_1(g_2 g_3)$ is the associator of $\mathcal{C}^2$, and we dropped the subscripts $A_?$ on $\otimes$ to avoid clutter.

Condition (A3) says that the maps $g \mapsto E_g$ and $b \mapsto U_b$ form a functor; (A4) says that the maps ω_{g_1,g_2} are natural with respect to bigons; (A1) says that our functor is strictly unital; (A2) is equivalent to the coherence conditions [29, (4.3)] for unitors; (A5) is [29, (4.2)].

To define the equivariant Brauer group of $\mathcal{C}$, we also need equivalences between such actions. Let $(A^1, E^1, \omega^1, U^1)$ and $(A^2, E^2, \omega^2, U^2)$ be continuous actions of $\mathcal{C}$. A *transformation* between them consists of the following data:

- a $C_0(\mathcal{C}^0)$-linear correspondence F between A^1 and A^2, with fibres F_x for $x \in \mathcal{C}^0$;
- isomorphisms of correspondences

  $$V_g \colon E^1_g \otimes_{A^1_{s(g)}} F_{s(g)} \to F_{r(g)} \otimes_{A^2_{r(g)}} E^2_g$$

  for all $g \in \mathcal{C}^1$, which are continuous in the sense that they give an isomorphism $V \colon E^1 \otimes_{s^*(A^1)} s^*(F) \to r^*(F) \otimes_{r^*(A^2)} E^2$;

this must satisfy the following conditions:

(T1) for each bigon $b \colon g \Rightarrow h$, the following diagram commutes:

$$\begin{tikzcd}
E^1_g \otimes_{A^1_{s(g)}} F_{s(g)} \arrow[r, "V_g"] \arrow[d, "U^1_b \otimes 1"'] & F_{r(g)} \otimes_{A^2_{r(g)}} E^2_g \arrow[d, "1 \otimes U^2_b"] \\
E^1_h \otimes_{A^1_{s(h)}} F_{s(h)} \arrow[r, "V_h"] & F_{r(h)} \otimes_{A^2_{r(h)}} E^2_h
\end{tikzcd}$$

(T2) $V_{1_x} \colon A^1_x \otimes_{A^1_x} F_x \to F_x \otimes_{A^2_x} A^2_x$ is the canonical isomorphism;

(T3) the following diagram commutes for $g, h \in \mathcal{C}^1$ with $s(g) = r(h)$:

$$\begin{tikzcd}
E^1_g \otimes_{A^1_{s(g)}} E^1_h \otimes_{A^1_{s(h)}} F_{s(h)} \arrow[r, "\omega^1_{g,h} \otimes 1"] \arrow[dd, "1 \otimes V_h"'] & E^1_{gh} \otimes_{A^1_{s(h)}} F_{s(h)} \arrow[d, "V_{gh}"] \\
 & F_{r(g)} \otimes_{A^2_{r(g)}} E^2_{gh} \\
E^1_g \otimes_{A^1_{s(g)}} F_{s(g)} \otimes_{A^2_{s(g)}} E^2_h \arrow[r, "V_g \otimes 1"] & F_{r(g)} \otimes_{A^2_{r(g)}} E^2_g \otimes_{A^2_{s(g)}} E^2_h \arrow[u, "1 \otimes \omega^2_{g,h}"']
\end{tikzcd}$$

Condition (T1) says that V is natural with respect to bigons, (T2) is the coherence condition for units and (T3) is the coherence condition for multiplication.

An *equivalence* between two actions is a transformation where each F_x or, equivalently, F is an imprimitivity bimodule; then the maps V_g and V are automatically isomorphisms of imprimitivity bimodules, that is, compatibility with the left inner product comes for free.

A *modification* between two transformations

$$(F^1, V^1), (F^2, V^2) \colon (A^1, E^1, \omega^1, U^1) \rightrightarrows (A^2, E^2, \omega^2, U^2)$$

is given by the following data:

- isomorphisms of correspondences $W_x \colon F_x^1 \to F_x^2$ for all $x \in \mathcal{C}^0$ that give an isomorphism $W \colon F^1 \to F^2$;

this must satisfy the following condition:

(M1) for each $g \in \mathcal{C}^1$ the following diagram commutes:

$$\begin{tikzcd}
E_g^1 \otimes_{A^1_{s(g)}} F^1_{s(g)} \arrow[r, "V_g^1"] \arrow[d, "1\otimes W_{s(g)}"'] & F^1_{r(g)} \otimes_{A^2_{r(g)}} E_g^2 \arrow[d, "W_{r(g)}\otimes 1"] \\
E_g^1 \otimes_{A^1_{s(g)}} F^2_{s(g)} \arrow[r, "V_g^2"] & F^2_{r(g)} \otimes_{A^2_{r(g)}} E_g^2
\end{tikzcd}$$

This is the obvious notion of isomorphism between two transformations.

Two actions of $\mathcal{C}$ may be tensored together in the obvious way, using the fibrewise maximal tensor product as in [4].

Definition 7.3.1. The *equivariant Brauer group* $\mathrm{Br}(\mathcal{C})$ of the locally compact bigroupoid $\mathcal{C}$ is the set of equivalence classes of continuous actions of $\mathcal{C}$ on continuous trace C^*-algebras with spectrum $\mathcal{C}^0$; the group structure is the tensor product over $\mathcal{C}^0$. We also write $\mathrm{Br}_{\mathcal{G}}(P) = \mathrm{Br}(\mathcal{G} \ltimes P)$.

The usual proof in [36] that this defines an Abelian group carries over to our case. The formula for the multiplication is particularly easy:

$$(A_1, E_1, \omega_1, U_1) \otimes_P (A_2, E_2, \omega_2, U_2) = (A_1 \otimes_P A_2, E_1 \otimes_P E_2, \omega_1 \otimes_P \omega_2, U_1 \otimes_P U_2),$$

where $\otimes_P$ means the maximal fibrewise tensor product of C^*-algebras over P, the corresponding external tensor product of imprimitivity bimodules, or the external tensor product of operators. Similar formulas work for equivalences between actions, so that the operation $\otimes_P$ descends to equivalence classes. The usual symmetry of $\otimes_P$ gives that this multiplication is commutative. Further details are left to the reader.

Definition 7.3.2. The *equivariant Picard group* $\mathrm{Pic}_{\mathcal{C}}(A, E, \omega, U)$ of an action of $\mathcal{C}$ is the group of all equivalence classes of self-equivalences on (A, E, ω, U), where two self-equivalences are considered equivalent if there is a modification between them. The group structure is the (vertical) composition of transformations.

In the special case of actions on continuous trace C^*-algebras, the definitions above simplify because of the following lemma:

Lemma 7.3.3. *Let A and B be continuous trace C^*-algebras with spectrum X. Let E_1 and E_2 be two $C_0(X)$-linear equivalences from A to B. There is a complex line bundle L over X with $E_2 \cong E_1 \otimes_X L$, and conversely $E_1 \otimes_X L$ is another equivalence from A to B for any complex line bundle L.*

Let $U_1, U_2 \colon E_1 \rightrightarrows E_2$ be two isomorphisms of equivalences. There is a continuous map $\varphi \colon X \to \mathbb{T}$ with $U_2(x) = \varphi(x) \cdot U_1(x)$ for all $x \in X$, and conversely $U_1 \cdot \varphi$ for a continuous map $\varphi \colon X \to \mathbb{T}$ is another isomorphism $E_1 \to E_2$.

We always identify a complex line bundle with its space of C_0-sections, which is a $C_0(P), C_0(P)$-imprimitivity bimodule.

Proof. Let E_2^* be the inverse equivalence. Then $E_1 \otimes_B E_2^*$ is a $C_0(X)$-linear self-equivalence of A. So we have to prove that any $C_0(X)$-linear self-equivalence of A is of the form $A \otimes_X L$ for a complex line bundle L over X. The opposite algebra A^{op} is an inverse for A in the Brauer group, that is, $A \otimes_X A^{\mathrm{op}} \cong C_0(X, \mathbb{K})$; this is Morita equivalent to $C_0(X)$. It is well-known that a $C_0(X)$-linear self-equivalence of $C_0(X)$ is the same as a complex line bundle over X. Since $C_0(X)$ and $C_0(X, \mathbb{K})$ are $C_0(X)$-linearly equivalent, they have isomorphic groups of $C_0(X)$-linear self-equivalences. Thus $E \otimes_X A^{\mathrm{op}}$ is the space $L \otimes \mathbb{K}$, where L is the space of sections of a complex line bundle over X. On the one hand, $E \otimes_X (A^{\mathrm{op}} \otimes_X A) \cong E \otimes_X C_0(X, \mathbb{K})$ is just the stabilisation of E; on the other hand, it is $(E \otimes_X A^{\mathrm{op}}) \otimes_X A \cong L \otimes \mathbb{K} \otimes_X A$. Due to the $C_0(X)$-linear equivalence between A and $A \otimes \mathbb{K}$, we may remove the stabilisations again to see that $E \cong A \otimes_X L$. It is clear, conversely, that $E \otimes_X L$ is again a $C_0(X)$-linear self-equivalence of A.

If $f_1, f_2 \colon E_1 \to E_2$ are two isomorphisms of imprimitivity bimodules, then $f_2^{-1} f_1 \colon E_1 \to E_1$ is an isomorphism. In each fibre, $(E_1)_x$ is an imprimitivity bimodule from $\mathbb{K}$ to $\mathbb{K}$. The isomorphism $f_2^{-1} f_1$ gives a unitary bimodule map, and any such map is just multiplication with a scalar of absolute value 1. This gives the function $\varphi \colon X \to \mathbb{T}$, which is continuous because U_1 and U_2 are continuous. $\square$

Proposition 7.3.4. *The groups $\mathrm{Pic}_{\mathcal{C}}(A, E, \omega, U)$ are canonically isomorphic for all actions (A, E, ω, U) of $\mathcal{C}$ on continuous-trace C^*-algebras over $\mathcal{C}^0$.*

Proof. Let $(A_i, E_i, \omega_i, U_i)$ for $i = 1, 2$ be two actions of $\mathcal{C}$ on continuous-trace C^*-algebras over $\mathcal{C}^0$. Since the Brauer group has inverses, there are other actions $(A_i, E_i, \omega_i, U_i)$, $i = 3, 4$, of $\mathcal{C}$ on continuous-trace C^*-algebras over $\mathcal{C}^0$ such that

$$(A_1, E_1, \omega_1, U_1) \otimes_{\mathcal{C}^0} (A_3, E_3, \omega_3, U_3) \cong (A_2, E_2, \omega_2, U_2),$$
$$(A_2, E_2, \omega_2, U_2) \otimes_{\mathcal{C}^0} (A_4, E_4, \omega_4, U_4) \cong (A_1, E_1, \omega_1, U_1).$$

Tensoring a self-equivalence of $(A_1, E_1, \omega_1, U_1)$ with the identity equivalence of $(A_3, E_3, \omega_3, U_3)$ gives a self-equivalence of $(A_2, E_2, \omega_2, U_2)$. Tensoring a self-equivalence of $(A_2, E_2, \omega_2, U_2)$ with the identity equivalence of $(A_4, E_4, \omega_4, U_4)$ gives a self-equivalence of $(A_1, E_1, \omega_1, U_1)$. This defines group homomorphisms between the two Picard groups that are inverse to each other because $(A_3, E_3, \omega_3, U_3)$ and $(A_4, E_4, \omega_4, U_4)$ are inverse to each other in the Brauer group. □

We denote the common Picard group of all actions of $\mathcal{C}$ on continuous-trace C^*-algebras over $\mathcal{C}^0$ by $\operatorname{Pic}(\mathcal{C})$ and also write $\operatorname{Pic}_{\mathcal{G}}(P) = \operatorname{Pic}(\mathcal{G} \ltimes P)$.

Lemma 7.3.3 becomes even more powerful when we use that the functors that send a paracompact space X to the set of equivalence classes of complex line bundles or continuous trace C^*-algebras with spectrum X are both homotopy invariant; actually, they are $H^2(X, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$.

Lemma 7.3.5. *Assume that the spaces $\mathcal{C}^i$ are paracompact. Assume that there is a continuous homotopy $H\colon \mathcal{C}^1 \times [0, 1] \to \mathcal{C}^1$ with $H_0 = \mathrm{id}_{\mathcal{C}^1}$, $H_1 = u \circ r$ for the unit and range maps u, r between $\mathcal{C}^1$ and $\mathcal{C}^0$, and $r \circ H_t = r$ for all $t \in [0, 1]$. Assume further that $r_2\colon \mathcal{C}^2 \to \mathcal{C}^1$ is a homotopy equivalence. Let A be any continuous trace C^*-algebra with spectrum $\mathcal{C}^0$. Then:*

(a) there is an imprimitivity bimodule E between $r^*(A)$ and $s^*(A)$ that restricts to the identity on units;

(b) any two such E are isomorphic with an isomorphism that is the identity on units;

(c) there are isomorphisms

$$\omega\colon \operatorname{pr}_1^*(E) \otimes_{(s\operatorname{pr}_1)^*(A)} \operatorname{pr}_2^*(E) \to \mu^*(E) \quad \text{and} \quad U\colon s_2^*(E) \to r_2^*(E)$$

such that $\omega_{1_{r(g)},g}$ and $\omega_{g,1_{s(g)}}$ are the canonical isomorphisms and U_{1_g} is the identity for all $g \in \mathcal{C}^1$;

(d) any two choices for ω and U as above differ by pointwise multiplication with $\exp(2\pi\mathrm{i}\varphi)$ for a continuous map $\varphi\colon \mathcal{C}^1 \times_{s,\mathcal{C}^0,r} \mathcal{C}^1 \sqcup \mathcal{C}^2 \to \mathbb{R}$ with $\varphi(1_{r(g)}, g) = 0$, $\varphi(g, 1_{s(g)}) = 0$ and $\varphi(1_g) = 0$ for all $g \in \mathcal{C}^1$.

Let $(A^i, E^i, \omega^i, U^i)$ for $i = 1, 2$ be actions of $\mathcal{C}$ and let F be an A^1, A^2-imprimitivity bimodule. Then:

(i) there is an isomorphism of imprimitivity bimodules $V\colon E^1 \otimes_{s^*(A^1)} s^*(F) \cong r^*(F) \otimes_{r^*(A^2)} E^2$ that restricts to the canonical isomorphism on units;

(ii) any two choices V^1 and V^2 as in (i) differ by pointwise multiplication with $\exp(2\pi \mathrm{i}\varphi)$ for a continuous map $\varphi\colon \mathcal{C}^1 \to \mathbb{R}$ with $\varphi(1_x) = 0$ for all $x \in \mathcal{C}^0$.

Proof. The pull-backs $s^*(A)$ and $r^*(A)$ are continuous trace C^*-algebras over $\mathcal{C}^1$ that restrict to the same continuous trace C^*-algebra on $u(\mathcal{C}^0) \subseteq \mathcal{C}^1$. By assumption, $\mathcal{C}^0$ is a deformation retract of $\mathcal{C}^1$, and the functor $X \mapsto \mathrm{Br}(X)$ is homotopy invariant on paracompact spaces. Since $u^*s^*(A) \cong u^*r^*(A)$, there must be an imprimitivity bimodule E between $s^*(A)$ and $r^*(A)$. The restriction of E to units differs from the canonical one by some line bundle by Lemma 7.3.3. This line bundle extends to a line bundle over $\mathcal{C}^1$ because $\mathcal{C}^0$ is a deformation retract of $\mathcal{C}^1$. Tensoring E with the opposite of that line bundle, we can arrange that $u^*(E)$ is isomorphic to the identity equivalence on A; then we may replace E by an isomorphic equivalence such that $u^*(E)$ is *equal* and not just isomorphic to the identity equivalence on A. This proves (a).

Let E^1 and E^2 be two equivalences between $s^*(A)$ and $r^*(A)$ that restrict to the identity on units. Then $E^2 \cong E^1 \otimes_{\mathcal{C}^1} L$ for some line bundle L by Lemma 7.3.3. Since $u^*(E^2) = u^*(E^1)$, $u^*(L)$ is trivialisable. Since $\mathcal{C}^0$ is a deformation retract of $\mathcal{C}^1$, this implies that L is trivialisable over $\mathcal{C}^1$, so $E^1 \cong E^2$. This isomorphism restricted to units differs from the identity isomorphism by pointwise multiplication with some function $\psi\colon \mathcal{C}^0 \to \mathbb{T}$ by Lemma 7.3.3. Since the unit map is a homotopy equivalence, this function extends to $\bar{\psi}\colon \mathcal{C}^1 \to \mathbb{T}$. Multiplying pointwise with $\bar{\psi}^{-1}$ gives another isomorphism $E^1 \cong E^2$ that restricts to the identity on units. This proves (b).

We claim that the inclusion of the subspace

$$X = \{(g, 1_{s(g)}) \mid g \in \mathcal{C}^1\} \cup \{(1_{r(h)}, h) \mid h \in \mathcal{C}^1\}$$

into $\mathcal{C}^1 \times_{s,\mathcal{C}^0,r} \mathcal{C}^1$ is a homotopy equivalence. Indeed, the maps $(g, h) \mapsto (g, H_t(h))$ deformation-retract $\mathcal{C}^1 \times_{s,\mathcal{C}^0,r} \mathcal{C}^1$ to the subspace of pairs $(g, 1_{s(g)})$, and they restrict to a deformation retraction from the subspace X onto the same space.

The $C_0(\mathcal{C}^1 \times_{s,r} \mathcal{C}^1)$-linear imprimitivity bimodules $\mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E)$ and $\mu^*(E)$ differ by some line bundle by Lemma 7.3.3. Since the two imprimitivity bimodules are canonically isomorphic on the subspace X, the line bundle is trivial on X. Since the inclusion of X is a homotopy equivalence, the line bundle is trivial everywhere. Hence there is an isomorphism ω between $\mathrm{pr}_1^*(E) \otimes_{(s\,\mathrm{pr}_1)^*(A)} \mathrm{pr}_2^*(E)$ and $\mu^*(E)$. On X, we also have the canonical isomorphism, which differs from ω by some continuous function $\psi\colon X \to \mathbb{T}$

by Lemma 7.3.3. As above, we may correct the isomorphism ω so that it restricts to the identity on X because ψ extends continuously to $\mathcal{C}^1 \times_{s,r} \mathcal{C}^1$.

A similar argument gives an isomorphism $U\colon s_2^*(E) \to r_2^*(E)$ over the space of bigons $\mathcal{C}^2$ with $U(1_g) = \mathrm{id}_{E_g}$ for all $g \in \mathcal{C}^1$ because r_2 is a homotopy equivalence. This proves (c).

Any two choices for ω and U differ through pointwise multiplication with some functions $\mathcal{C}^1 \times_{s,r} \mathcal{C}^1 \to \mathbb{T}$ and $\mathcal{C}^2 \to \mathbb{T}$ by Lemma 7.3.3; these functions are 1 on X or on unit bigons by our normalisations. Since the inclusions of X and unit bigons are homotopy equivalences, covering space theory allows to lift such a function to $\mathbb{R}$ as required in (d).

The proofs of (i) and (ii) use the same ideas. An isomorphism V exists because the two imprimitivity bimodules are isomorphic on units and the inclusion of units is a homotopy equivalence, and it may be arranged to be the canonical isomorphism on units because any continuous function $u(\mathcal{C}^0) \to \mathbb{T}$ extends to a continuous function $\mathcal{C}^1 \to \mathbb{T}$. Lemma 7.3.3 shows that two isomorphisms V_1 and V_2 differ by a function $\mathcal{C}^1 \to \mathbb{T}$, which is constant equal to 1 on units. Any such function lifts to $\mathbb{R}$, giving (ii). □

Let (A, E, ω, U) be a continuous action of $\mathcal{C}$. If F is a $C_0(\mathcal{C}^0)$-linear Morita equivalence from A to some other C^*-algebra A', then we may transfer the action from A to A' along F: let $E' = F \otimes_A E \otimes_A F^*$ and translate ω and U accordingly. Hence up to equivalence of actions, only the equivalence class of A matters. Similarly, if A is fixed and E' is another equivalence $s^*(A) \cong r^*(A)$ with $E \cong E'$, then we may use the isomorphism $E \cong E'$ to transfer ω and U to E'. So in the definition of the Brauer group, only the Morita equivalence class of A and, for fixed A, the isomorphism class of E matter.

7.4 Lifting actions to continuous trace algebras

Let P be a second countable locally compact space with a continuous action of $\mathbb{R}^n$; then P is paracompact. Lemma 7.3.5 applies to the transformation groupoid $\mathbb{R}^n \ltimes P$ and the transformation bigroupoid $\mathcal{G} \ltimes P$ because the Lie groups $\mathbb{R}^n$ and $\Lambda^3\mathbb{R}^n$ are contractible. We use this to analyse the obstruction to lifting an $\mathbb{R}^n$-action from P to a continuous trace C^*-algebra over P. In the case of $\mathbb{R}^n \ltimes P$, our results are also contained in the results of [36].

Consider the case $\mathbb{R}^n$ first. Here there are no bigons, so the datum U is not there and the conditions (A3)–(A4) in the definition of an action are empty. Let A be a continuous trace C^*-algebra over P. Lemma 7.3.5 provides the data E and ω for an action, satisfying (A1) and (A2), but not yet satisfying the cocycle condition (A5). Since E is unique up to isomorphism, its choice does not affect whether or not there is ω satisfying (A5), nor the resulting element in the equivariant Brauer group. By Lemma 7.3.5, any two choices for ω differ through pointwise multiplication with a function of the form $\exp(2\pi \mathrm{i}\varphi)$ for a continuous function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \times P \to \mathbb{R}$; here we have identified the space of composable arrows in $\mathbb{R}^n \ltimes P$ with $\mathbb{R}^n \times \mathbb{R}^n \times P$.

Similarly, the space of composable k-tuples of arrows in $\mathbb{R}^n \ltimes P$ is $(\mathbb{R}^n)^k \times P$. By Lemma 7.3.3, the two isomorphisms $(E_{g_1} \otimes E_{g_2}) \otimes E_{g_3} \to E_{g_1 g_2 g_3}$ in (A5) differ by pointwise multiplication with a function $(\mathbb{R}^n)^3 \times P \to \mathbb{T}$. We can also view this as the difference between the isomorphism $(E_{g_1} \otimes E_{g_2}) \otimes E_{g_3} \to E_{g_1} \otimes (E_{g_2} \otimes E_{g_3})$ induced by the two isomorphisms above and the canonical one, which makes it clear that this function plays the role of an associator. It is 1 if one of the $\mathbb{R}^n$-entries is 0 by our normalisations. Hence arguments as in the proof of Lemma 7.3.5 show that it lifts uniquely to an $\mathbb{R}$-valued function $\psi\colon (\mathbb{R}^n)^3 \times P \to \mathbb{R}$ that is 0 if one of the $\mathbb{R}^n$-entries is 0. When we multiply ω pointwise by $\exp(2\pi \mathrm{i} \varphi)$, then this adds the following function to ψ:

$$\partial\varphi(t_1, t_2, t_3, p) = \varphi(t_2, t_3, p) - \varphi(t_1 + t_2, t_3, p) + \varphi(t_1, t_2 + t_3, p) - \varphi(t_1, t_2, \alpha_{t_3}(p)).$$

Since the associator diagram

$$\begin{tikzcd}
((E_{g_1} \otimes E_{g_2}) \otimes E_{g_3}) \otimes E_{g_4} \arrow[r] \arrow[dd] & (E_{g_1} \otimes (E_{g_2} \otimes E_{g_3})) \otimes E_{g_4} \arrow[d] \\
 & E_{g_1} \otimes ((E_{g_2} \otimes E_{g_3}) \otimes E_{g_4}) \arrow[d] \\
(E_{g_1} \otimes E_{g_2}) \otimes (E_{g_3} \otimes E_{g_4}) \arrow[r] & E_{g_1} \otimes (E_{g_2} \otimes (E_{g_3} \otimes E_{g_4}))
\end{tikzcd}$$

commutes, we deduce that ψ automatically satisfies the cocycle condition

$$\begin{aligned} 0 = \partial\psi(t_1, t_2, t_3, t_4, p) &= \psi(t_2, t_3, t_4, p) - \psi(t_1 + t_2, t_3, t_4, p) \\ &\quad + \psi(t_1, t_2 + t_3, t_4, p) - \psi(t_1, t_2, t_3 + t_4, p) + \psi(t_1, t_2, t_3, \alpha_{t_4}(p)). \end{aligned}$$

More precisely, the function $\exp(2\pi \mathrm{i} \partial\psi)$ is automatically the constant function 1, and this implies the above because $\partial\psi$ vanishes if one of the t_i is 0 and the lifting of $\mathbb{T}$-valued functions to normalised $\mathbb{R}$-valued functions is unique if it exists.

As a result, when we view ψ as a function from $(\mathbb{R}^n)^3$ to the Fréchet space $C(P, \mathbb{R}^n)$ of continuous functions $P \to \mathbb{R}^n$ with the action of $\mathbb{R}^n$ induced from the action on P, then ψ is a continuous 3-cocycle; and we may choose ω to satisfy the cocycle condition (A5) if and only if this 3-cocycle is a coboundary. Thus the action of $\mathbb{R}^n$ on P lifts to an action on A if and only if the class of ψ in the continuous group cohomology $H^3_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ vanishes. We call this class the *lifting obstruction* of the continuous trace C^*-algebra A.

The Packer–Raeburn Stabilisation Trick shows that any action of $\mathbb{R}^n$ by equivalences as above is equivalent to a strict action by automorphisms on the stabilisation of A (this is contained in [29, Thm. 5.3]). Since stabilisation does not change the class of A in the Brauer group, we may assume that A is stable. Then the lifting obstruction is the obstruction to the existence of a strict action of $\mathbb{R}^n$ by automorphisms. An obstruction for this is also constructed in [36], but in the measurable group cohomology $H^3_{\mathrm{M}}(\mathbb{R}^n, C(P,\mathbb{R}))$. These two cohomology groups coincide by [149, Thm. 3].

The continuous cohomology group $H^3_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R}))$ may be simplified if the action of $\mathbb{R}^n$ factors through a torus $\mathbb{T}^n$. Let $\Omega^k\mathbb{R}^n = (\Lambda^k\mathbb{R}^n)^*$ denote the vector space of antisymmetric k-linear maps $(\mathbb{R}^n)^k \to \mathbb{R}$.

Proposition 7.4.1. *Let P be a second countable, locally compact $\mathbb{T}^n$-space, viewed as an $\mathbb{R}^n$-space, with orbit space $P/\mathbb{R}^n$. Then*

$$H^k_{\mathrm{cont}}(\mathbb{R}^n, C(P,\mathbb{R})) \cong C(P/\mathbb{R}^n, \Omega^k\mathbb{R}^n)$$

for all $k \geq 0$. The isomorphism maps $\chi\colon P/\mathbb{R}^n \to \Omega^k\mathbb{R}^n$ to the cocycle given by

$$(\mathbb{R}^n)^k \times P \to \mathbb{R}, \qquad (t_1, \dots, t_k, p) \mapsto \chi([p])(t_1 \wedge \dots \wedge t_k).$$

Proof. If the action of $\mathbb{T}^n$ on P is free, then this is [97, Lem. 2.1]. We explain why the result remains true for non-free actions of $\mathbb{T}^n$. Throughout this proof, group cohomology is understood to be continuous group cohomology.

Continuous representations of $\mathbb{R}^n$ on Fréchet spaces over $\mathbb{R}$ are equivalent to non-degenerate modules over the Banach algebra $L^1(\mathbb{R}^n)$ of integrable functions $\mathbb{R}^n \to \mathbb{R}$. Let $\mathbb{R}$ denote the trivial representation of $\mathbb{R}^n$. The continuous group cohomology for $\mathbb{R}^n$ with coefficients in a Fréchet $\mathbb{R}^n$-module W is the same as $\mathrm{Ext}^k_{L^1(\mathbb{R}^n)}(\mathbb{R}, W)$. Since $L^1(\mathbb{R}^n)$ is Abelian, the module structures on W and $\mathbb{R}$ induce an $L^1(\mathbb{R}^n)$-module structure on $\mathrm{Ext}^k_{L^1(\mathbb{R}^n)}(\mathbb{R}, W)$ as well, and both module structures are the same. Since one is trivial, so is the other.

Let V be a Fréchet space with a continuous action of $\mathbb{T}^n$, which we view as an action of $\mathbb{R}^n$. Split $V \cong V^1 \oplus V^2$ where $V^1 \subseteq V$ is the space of $\mathbb{T}^n$-invariant elements and V^2 is the closed linear span of the other homogeneous components. There is an element $f \in L^1(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ satisfies $\hat{f}(0) = 1$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}^n \setminus \{0\}$; for instance, we may take f to be the inverse Fourier transform of a smooth bump function around 0 supported in $(-1,1)^n$. The function f acts on V by the projection onto V^1. The induced action by f on the cohomology $H^k(\mathbb{R}^n, V)$ is the same as the action of $\hat{f}(0)$ because the latter is how f acts on the trivial representation. Hence $H^k(\mathbb{R}^n, V^2) = 0$, so $H^k(\mathbb{R}^n, V) = H^k(\mathbb{R}^n, V^1)$. In our case, the $\mathbb{T}^n$-invariant elements in $C(P,\mathbb{R})$ are exactly the functions in $C(P/\mathbb{R}^n, \mathbb{R})$.

This proves $H^k(\mathbb{R}^n, C(P,\mathbb{R})) \cong H^k(\mathbb{R}^n, C(P/\mathbb{R}^n, \mathbb{R}))$, where the action of $\mathbb{R}^n$ on the Fréchet space $C(P/\mathbb{R}^n, \mathbb{R})$ is trivial. Hence we get $H^k(\mathbb{R}^n, C(P/\mathbb{R}^n, \mathbb{R})) \cong H^k(\mathbb{R}^n, \mathbb{R}) \otimes_{\mathbb{R}}$

$C(P/\mathbb{R}^n, \mathbb{R})$, and $H^k(\mathbb{R}^n, \mathbb{R}) \cong \Omega^k\mathbb{R}^n$ as in the proof of [97, Lem. 2.1]. □

Remark 7.4.2. Proposition 7.4.1 still works if the stabiliser lattice $\mathbb{Z}^n$ of the action varies over P, that is, if there is a continuous function $\gamma\colon P/\mathbb{R}^n \to \mathrm{Gl}_n(\mathbb{R})/\mathrm{Gl}_n(\mathbb{Z})$ such that, for each $p \in P$, the lattice $\gamma_{[p]}(\mathbb{Z}^n) \subseteq \mathbb{R}^n$ fixes p.

Now we replace $\mathbb{R}^n$ by the crossed module $\mathcal{G}$. We assume that the canonical map

$$C(P/\mathbb{R}^n, \Omega^k\mathbb{R}) \cong H^k_{\mathrm{cont}}(\mathbb{R}^n, C(P/\mathbb{R}^n, \mathbb{R})) \to H^k_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$$

is an isomorphism for $k = 2, 3$. This holds, in particular, in the situation of Proposition 7.4.1 or Remark 7.4.2. Theorems 7.4.3–7.4.4 extend to this case, although we only state them in the situation of Proposition 7.4.1.

Under our assumption, the lifting obstruction in $H^3_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ is cohomologous to a *unique* function $\psi\colon P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$. We also call this function the *lifting obstruction* of A. The action of $\mathbb{R}^n$ on P lifts to an action on A if and only if this function $P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$ vanishes.

Theorem 7.4.3. *Let P be a second countable, locally compact $\mathbb{T}^n$-space and let A be a continuous trace C^*-algebra with spectrum P. Then the action of $\mathbb{T}^n$ on P lifts to an action of the bigroupoid $\mathcal{G} \ltimes P$ on A.*

Proof. Since $\mathcal{G} \ltimes P$ and $\mathbb{R}^n \ltimes P$ have the same objects and arrows, we may construct E and ω as above. For an action of $\mathcal{G} \ltimes P$, we also need the datum U, and this modifies the cocycle condition for ω.

The condition (A3) for vertical products says that $U(\xi, t, p)$ for fixed $t \in \mathbb{R}^n$ and $p \in P$ is a continuous homomorphism $\Lambda^3\mathbb{R}^n \to \mathbb{T}$. In particular, $U(0, t, p) = 1$ for all t, p. The condition (A4) for horizontal products gives

$$U(\xi_1, t_1, \alpha_{t_2}(p_2)) \cdot U(\xi_2, t_2, p_2) = U(\xi_1 + \xi_2, t_1 + t_2, p_2)$$

for all $\xi_1, \xi_2 \in \Lambda^3\mathbb{R}^n$, $t_1, t_2 \in \mathbb{R}^n$ and $p \in P$. For $t_2 = 0$ and $\xi_1 = 0$, this says that $U(\xi_2, 0, p_2) = U(\xi_2, t_1, p_2)$, so $U(\xi, t, p)$ does not depend on t and we may write it $U(\xi, p)$; for $t_1 = 0$ and $\xi_2 = 0$, it says $U(\xi_1, \alpha_{t_2}(p_2)) = U(\xi_1, p_2)$, that is, $U(\xi, p)$ only depends on the $\mathbb{R}^n$-orbit $[p]$ of p. For a function depending only on ξ and $[p]$, the two multiplicativity conditions (A3) and (A4) are equivalent. Thus (A3) and (A4) say exactly that U is a continuous function from $P/\mathbb{R}^n$ to the space of group homomorphisms $\Lambda^3\mathbb{R}^n \to \mathbb{T}$. Such homomorphisms lift uniquely to homomorphisms $\Lambda^3\mathbb{R}^n \to \mathbb{R}$. Thus $U(\xi, t, p) = \exp(2\pi\mathrm{i}\upsilon(\xi, [p]))$ for a continuous function $\upsilon\colon P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$ that is uniquely determined by U.

Proposition 7.4.1 shows that there is a unique continuous function $P/\mathbb{R}^n \to \Omega^3\mathbb{R}^n$ that represents the lifting obstruction for an $\mathbb{R}^n$-action; that is, this function measures the failure of the cocycle condition (A5), without the associator U, for a particular choice of ω. When we put in the associator, then (A5) holds if and only if υ above is equal to the lifting obstruction. Hence there is an action (E, ω, U) of $\mathcal{G} \ltimes P$ on A. $\square$

Our next goal is a long exact sequence containing the Brauer and Picard groups of $\mathcal{C}$ and some known cohomology groups of P and $P/\mathbb{R}^n$. Let $\mathrm{For}\colon \mathrm{Br}_{\mathcal{G}}(P) \to \mathrm{Br}(P)$ and $\mathrm{For}\colon \mathrm{Pic}_{\mathcal{G}}(P) \to \mathrm{Pic}(P)$ denote the forgetful maps.

Theorem 7.4.4. *Let P be a second countable, locally compact $\mathbb{T}^n$-space. There is a natural long exact sequence*

$$\begin{aligned}0 \leftarrow \mathrm{Br}(P) \xleftarrow{\mathrm{For}} \mathrm{Br}_{\mathcal{G}}(P) \leftarrow C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n) \leftarrow \mathrm{Pic}(P) \xleftarrow{\mathrm{For}} \mathrm{Pic}_{\mathcal{G}}(P)& \\ \leftarrow C(P/\mathbb{R}^n, \mathbb{R}^n) \leftarrow H^1(P, \mathbb{Z}) \leftarrow H^1(P/\mathbb{R}^n, \mathbb{Z}) \leftarrow 0.&\end{aligned}$$

Here $H^1(-, \mathbb{Z})$ denotes Čech cohomology. Furthermore, there are natural isomorphisms $\mathrm{Br}(P) \cong H^3(P, \mathbb{Z})$ *and* $\mathrm{Pic}(P) \cong H^2(P, \mathbb{Z})$.

Proof. The surjectivity of $\mathrm{For}\colon \mathrm{Br}_{\mathcal{G}}(P) \to \mathrm{Br}(P)$ is asserted in Theorem 7.4.3. Choose such an action (E, ω, U). Any other action (E', ω', U') has $E' \cong E$ by (b) in Lemma 7.3.5; this gives an equivalence to an action with $E' = E$. The proof of Theorem 7.4.3 shows that we cannot modify U at all, that is, $U = U'$, because it must be the exponential of the lifting obstruction of A. The freedom in the choice of ω is to multiply it with $\exp(2\pi \mathrm{i} \varphi)$ for a continuous function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \times P \to \mathbb{R}$ normalised by $\varphi(t, 0, p) = \varphi(0, t, p) = 0$; it must satisfy $\partial\varphi = 0$ so as not to violate (A5). That is, φ is a cocycle for $H^2_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$.

An equivalence of actions allows, among other things, to conjugate ω by an isomorphism $V\colon E \to E$ that restricts to the identity on units. By Lemma 7.3.5, this function V differs from the identity by a continuous function $\mathbb{R}^n \times P \to \mathbb{T}$, which lifts uniquely to a continuous function $\kappa\colon \mathbb{R}^n \times P \to \mathbb{R}$ normalised by $\kappa(0, p) = 0$ for all $p \in P$. Conjugating ω by the equivalence of actions defined by κ multiplies it by $\exp(2\pi \mathrm{i} \partial\kappa)$ with

$$\partial\kappa(t_1, t_2, p) = \kappa(t_2, p) - \kappa(t_1 + t_2, p) + \kappa(t_1, \alpha_{t_2}(p)).$$

Thus only the class of φ in $H^2_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ matters for the equivalence class of the action. Proposition 7.4.1 shows that this cohomology group is $C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$, that is, any cocycle φ is cohomologous to a unique one of the form $(t_1, t_2, p) \mapsto \chi[p](t_1 \wedge t_2)$ for a continuous function $\chi\colon P/\mathbb{R}^n \to \Omega^2\mathbb{R}^n$. Summing up, any action of $\mathcal{C}$ on A is equivalent to one having the same E and U and $\omega \cdot \exp(2\pi \mathrm{i}\chi)$ for some $\chi \in C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$. Twisting the unit element of $\mathrm{Br}_{\mathcal{G}}(P)$ with $\exp(2\pi \mathrm{i}\chi)$ as above defines a group homomorphism $C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n) \to \mathrm{Br}_{\mathcal{G}}(P)$, which is one of the maps in our exact sequence. Our argument shows that its range is the kernel of $\mathrm{For}\colon \mathrm{Br}_{\mathcal{G}}(P) \to \mathrm{Br}(P)$, that is, our sequence is exact at $\mathrm{Br}_{\mathcal{G}}(P)$.

So far, we have only used equivalences of a special form. In general, an equivalence between the two actions of $\mathcal{C}$ on A also involves a self-equivalence of A, that is, an element $F \in \operatorname{Pic}(P)$. This is given by a line bundle over P by Lemma 7.3.3. An equivalence $A \sim A'$ allows to transport the given action (E, ω, U) on A to an action (E', ω', U') on A'; now we apply this to the self-equivalence F. This gives $E' = r^*(L) \otimes_P E \otimes_P s^*(L)^*$, with ω' and U' defined by first cancelling pull-backs of $L \otimes_P L^*$ in the middle and then applying ω and U. By Lemma 7.3.5 (b), there is an isomorphism $E' \cong E$ that restricts to the identity on units. Using this, we transfer ω' and U' to E. The argument above shows that, choosing the part V in the equivalence suitably, we may arrange that $\omega' = \exp(2\pi \mathrm{i}\chi)\omega$ and $U' = U$ for some $\chi \in C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; furthermore, χ is independent of choices.

Sending $F \in \operatorname{Pic}(P)$ to this $\chi \in C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$ gives a well-defined map $\operatorname{Pic}(P) \to C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; this is the next map in our exact sequence. It is routine to check that this map is a group homomorphism. Since we have now used the most general form of an equivalence, the actions (E, ω, U) and $(E, \exp(2\pi\mathrm{i}\chi)\omega, U)$ are equivalent if and only if χ is in the image of $\operatorname{Pic}(P) \to C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$; that is, our sequence is exact at $C(P/\mathbb{R}^n, \Omega^2\mathbb{R}^n)$. Furthermore, if $\chi = 0$ then the equivalence (F, V) from (E, ω, U) to the twist by χ is a *self*-equivalence, so we have lifted $F \in \operatorname{Pic}(P)$ to $(F, V) \in \operatorname{Pic}_{\mathcal{G}}(P)$. Thus our sequence is exact at $\operatorname{Pic}(P)$.

Now we consider the kernel of $\mathrm{For}\colon \operatorname{Pic}_{\mathcal{G}}(P) \to \operatorname{Pic}(P)$; this consists of self-equivalences (F, V) of (A, E, ω, U) where F is isomorphic to the identity equivalence on A; we may arrange $F = A$ by a modification. Any two choices for V differ by pointwise multiplication with $\exp(2\pi\mathrm{i}\varphi)$ for some continuous function $\varphi\colon \mathbb{R}^n \times P \to \mathbb{R}$ normalised by $\varphi(0, p) = 0$ for all $p \in P$ by (ii) in Lemma 7.3.5. Since V already satisfies the coherence condition (T3) for a cocycle, φ must be a cocycle for $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$.

Among the modifications from (F, V) to (F, V') for some V', we may consider those given by $W = \exp(2\pi\mathrm{i}\psi)$ for some $\psi\colon P \to \mathbb{R}$. This gives a modification from (F, V) to $(F, V \cdot \partial W)$ with $\partial W(t, p) = W(\alpha_t p) W(p)^{-1}$. Thus only the class of the cocycle ψ in $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R}))$ matters for the class in the Picard group. Proposition 7.4.1 identifies $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R})) \cong C(P/\mathbb{R}^n, \mathbb{R}^n)$. So we get a surjective group homomorphism from $C(P/\mathbb{R}^n, \mathbb{R}^n)$ onto the kernel of $\mathrm{For}\colon \operatorname{Pic}_{\mathcal{G}}(P) \to \operatorname{Pic}(P)$. This map continues our exact sequence and gives the exactness at $\operatorname{Pic}_{\mathcal{G}}(P)$.

So far, we have only used modifications that lift to a map $P \to \mathbb{R}$; general modifications involve continuous maps $\phi\colon P \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Locally, such a map lifts to $\mathbb{R}$. Choosing such a local lifting gives an open covering of P and a subordinate Čech 1-cocycle on P with values in $\mathbb{Z}$. There is a global lifting of ϕ to a function $P \to \mathbb{R}$ if and only if this 1-cocycle is a coboundary. Any Čech 1-cocycle in $H^1(P, \mathbb{Z})$ is the lifting obstruction of some continuous maps $\phi\colon P \to \mathbb{T}$ by a partition of unity argument. Thus $H^1(P, \mathbb{Z})$ is the quotient of the group of all functions $P \to \mathbb{T}$ modulo those functions of the form $\exp(2\pi\mathrm{i}\psi)$ for a continuous function $\psi\colon P \to \mathbb{R}$.

Any function $W\colon P \to \mathbb{T}$ gives a modification between a given transformation (F, V) and another transformation $(F, V \cdot \partial W)$ with ∂W as above. Since this has the same underlying equivalence F, ∂W is of the form $\partial W = \exp(2\pi i\kappa)$ for a continuous map $\kappa\colon \mathbb{R}^n \times P \to \mathbb{R}$, which represents a 1-cocycle in $H^1_{\mathrm{cont}}(\mathbb{R}^n, C(P, \mathbb{R})) \cong C(P/\mathbb{R}^n, \mathbb{R}^n)$. Thus there is a unique continuous function $\chi \in C(P/\mathbb{R}^n, \mathbb{R}^n)$ and a function $h\colon P \to \mathbb{R}$ such $\partial W = \exp(2\pi\mathrm{i}(\chi + \partial h))$. Hence we get a well-defined map $H^1(P, \mathbb{Z}) \to C(P/\mathbb{R}^n, \mathbb{R}^n)$ by sending the class of W in $H^1(P, \mathbb{Z})$ to the unique χ above. The image of this is the set of all χ for which $(F, V \exp(2\pi\mathrm{i}\chi))$ is equivalent to (F, V). This gives the exactness of our sequence at $C(P/\mathbb{R}^n, \mathbb{R}^n)$. Furthermore, $\chi = 0$ means that $\partial(W/h) = 1$ for some $h \in C(P, \mathbb{R})$. The condition $\partial(W/h) = 1$ says that W/h is a $\mathbb{R}^n$-invariant function on P, that is, $[W] \in H^1(P, \mathbb{Z})$ is in the image of $H^1(P/\mathbb{R}^n, \mathbb{Z})$; here the map $H^1(P/\mathbb{R}^n, \mathbb{Z}) \to H^1(P, \mathbb{Z})$ is induced by the quotient map $P \to P/\mathbb{R}^n$. We have shown exactness of our sequence at $H^1(P, \mathbb{Z})$.

If $\varphi\colon P/\mathbb{R}^n \to \mathbb{T}$ goes to the trivial element of $H^1(P, \mathbb{Z})$, then this means that $\varphi = \exp(2\pi\mathrm{i}\psi)$ for some $\psi\colon P \to \mathbb{R}$. Since φ is constant on $\mathbb{R}^n$-orbits and these are connected, ψ is also constant on $\mathbb{R}^n$-orbits, so we have lifted φ to $\psi\colon P/\mathbb{R}^n \to \mathbb{R}$. This means that φ gives the trivial element of $H^1(P/\mathbb{R}^n, \mathbb{Z})$; that is, our sequence is exact also at $H^1(P/\mathbb{R}^n, \mathbb{Z})$. The natural isomorphisms $\mathrm{Br}(P) \cong H^3(P, \mathbb{Z})$ and $\mathrm{Pic}(P) \cong H^2(P, \mathbb{Z})$ are well-known. $\square$

7.4.1 Non-associative algebras

We now interpret actions of the crossed module $\mathcal{G}$ as non-associative Fell bundles over the group $\mathbb{R}^n$. Taking the "section algebra" of such a non-associative Fell bundle gives the non-associative algebras of [22].

Let (A, E, ω, U) be an action of $\mathcal{G}$ on a C^*-algebra. This consists of imprimitivity A,A-bimodules E_t for $t \in \mathbb{R}^n$ with a continuity structure E and with a continuous multiplication map $\omega\colon \bigsqcup E_t \otimes_A E_u \to \bigsqcup E_{tu}$, and with a homomorphism $\Lambda^3\mathbb{R}^n \to ZUM(A)$, $\xi \mapsto U_\xi$, to the group $ZUM(A)$ of central unitary multipliers of A; here we use the same arguments as in the proof of Theorem 7.4.3 to see that the isomorphism $E_t \to E_t$ associated to the bigon $\xi\colon t \Rightarrow t$ does not depend on t, belongs to the centre, and that the map $\xi \mapsto U_\xi$ is a group homomorphism. In addition, these arguments show that the unitaries U_ξ are "invariant" under the $\mathbb{R}^n$-action; this means here that left and right multiplication by U_ξ on E_t gives the same map for each t; this is more than being in the centre of A, it means being in the centre of the whole Fell bundle.

We view the maps ω as a multiplication in a generalised Fell bundle. The cocycle condition for ω says that

$$x_t \cdot (x_u \cdot x_v) = U_{t\wedge u\wedge v} \cdot (x_t \cdot x_u) \cdot x_v$$

for all $x_t \in E_t$, $x_u \in E_u$, $x_v \in E_v$, $t, u, v \in \mathbb{R}^n$. Thus the multiplication in our generalised Fell bundle is not associative, but in a controlled fashion given by U. We do not have a Fell bundle over $\mathbb{R}^n$ unless U is trivial. The section C^*-algebra of an ordinary Fell bundle over $\mathbb{R}^n$ plays the role of the crossed product with $\mathbb{R}^n$ for ordinary actions by automorphisms. For a non-associative Fell bundle as above, we may still consider the space of compactly supported continuous sections of E and define a convolution product

$$(f_1 * f_2)(t) = \int_{\mathbb{R}^n} f_1(x) f_2(t - x)\, \mathrm{d}x.$$

The convolution product is, however, no longer associative. The non-associative algebras in [22] are of this form. It is not clear which analogue of a C^*-norm one could use on such a non-associative convolution algebra.

7.4.2 Making the action strict

So far, our actions on continuous trace C^*-algebras are actions by equivalences of the Lie bigroupoid $\mathcal{G}$. We are going to turn these into strict actions of the crossed module $\mathcal{H}$. First, the equivalence between $\mathcal{H}$ and $\mathcal{G}$ shows that actions of $\mathcal{G}$ and $\mathcal{H}$ are "equivalent." In particular, $\mathrm{Br}_{\mathcal{G}}(P) \cong \mathrm{Br}_{\mathcal{H}}(P)$. Secondly, any action by correspondences of the crossed module $\mathcal{H}$ is equivalent to a strict action by automorphisms, and equivalences among such actions are equivariant Morita equivalences in an almost classical sense.

Let (A, E, ω, U) be an action of $\mathcal{G} \ltimes P$ by correspondences. This is a morphism from the bicategory $\mathcal{G} \ltimes P$ to the correspondence bicategory, satisfying some continuity conditions. We may compose this with the morphism $\mathcal{C}_{\mathcal{H}} \ltimes P \to \mathcal{G} \ltimes P$ induced by $\pi\colon \mathcal{C}_{\mathcal{H}} \to \mathcal{G}$; this gives an action of $\mathcal{C}_{\mathcal{H}} \ltimes P$ by correspondences. Conversely, an action of $\mathcal{C}_{\mathcal{H}} \ltimes P$ gives an action of $\mathcal{G} \ltimes P$ by composing with the morphism induced by $\iota\colon \mathcal{G} \to \mathcal{C}_{\mathcal{H}}$. Going back and forth induces a bijection between equivalence classes of actions because ι and π are inverse to each other up to equivalence. All this follows from general bicategory theory, which tells us how to compose morphisms between bicategories, how to compose transformations between such morphisms vertically and horizontally, and that there are related multiplication operations on modifications; in brief, bicategories with morphisms, transformations and modifications form a tricategory.

Everything above also works for topological bicategories and continuous morphisms, transformations and modifications. The correspondence bicategory, however, is not a topological bicategory in the usual sense. The continuity of a map from a locally compact space X to the set of C^*-algebras is defined in an *ad hoc* way by giving a $C_0(X)$-C^*-algebra as an extra datum. So continuity is not a property of a map, but extra structure. Consider a continuous map $f\colon Y \to X$ between locally compact spaces and a continuous map from X

to C^*-algebras, given by a map $x \mapsto A_x$ to C^*-algebras and a $C_0(X)$-C^*-algebra A with fibres A_x. Then the pull-back $f^*(A) = C_0(Y) \otimes_{C_0(X)} A$ is a $C_0(Y)$-C^*-algebra with fibres $A_{f(y)}$; this is how we compose a continuous map to C^*-algebras with the continuous map f. Similarly, we may pull back C^*-correspondences along continuous maps, and operators between them. This is how we compose continuous maps from locally compact spaces to the arrows and bigons in the correspondence bicategory. The general theory implies, in particular:

Theorem 7.4.5. *Let P be a locally compact $\mathbb{R}^n$-space. The morphisms π and ι between $\mathcal{G}$ and $\mathcal{H}$ induce an isomorphism $\mathrm{Br}_{\mathcal{G}}(P) \cong \mathrm{Br}_{\mathcal{H}}(P)$. If P is second countable and the $\mathbb{R}^n$ action factors through $\mathbb{T}^n$, then any continuous trace C^*-algebra over P carries an action of $\mathcal{H}$ lifting the action of $\mathbb{R}^n$ on P.*

Let (A', E', ω', U') and (A, E, ω, U) be actions of $\mathcal{H}$ and $\mathcal{G}$ corresponding to each other. Then the equivariant Picard groups $\mathrm{Pic}_{\mathcal{G}}(A, E, \omega, U)$ and $\mathrm{Pic}_{\mathcal{H}}(A', E', \omega', U')$ are canonically isomorphic. □

We also use the notation $\mathcal{H} \ltimes P$ for the bigroupoid $\mathcal{C}_{\mathcal{H}} \ltimes P$. Now we make explicit how a continuous action (A, E, ω, U) of $\mathcal{G} \ltimes P$ gives a continuous action (A', E', ω', U') of $\mathcal{H} \ltimes P$. We put $A' = A$, and $E'_{(t,\eta)} = E_t$ with $E' = \mathrm{pr}_1^*(E)$, where $\mathrm{pr}_1\colon \mathbb{R}^n \times \Lambda^2\mathbb{R}^n \to \mathbb{R}^n$ maps $(t, \eta) \mapsto t$. Moreover, $U'_{(\theta,\xi)} = U_\xi$ for all $\theta \in \Lambda^2\mathbb{R}^n$, $\xi \in \Lambda^3\mathbb{R}^n$. The map ω' is defined by composing

$$\begin{aligned} E'_{(t_1,\eta_1,\alpha_{t_2}(p_2))} \otimes E'_{(t_2,\eta_2,p_2)} = E_{(t_1,\alpha_{t_2}(p_2))} \otimes E_{(t_2,p_2)} &\xrightarrow{\omega(t_1,t_2,p_2)} E_{(t_1+t_2,p_2)} \\ &\xrightarrow{U(\omega_\pi((t_1,\eta_1),(t_2,\eta_2)))} E_{(t_1+t_2,p_2)} = E'_{(t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2,p_2)}, \end{aligned}$$

where $\omega_\pi((t_1, \eta_1), (t_2, \eta_2)) = t_1 \wedge \eta_2$ and where the tensors are over $A_{\alpha_{t_2}(p_2)}$. Thus

$$\omega'((t_1, \eta_1), (t_2, \eta_2, p_2)) = U(t_1 \wedge \eta_2, p_2) \cdot \omega(t_1, t_2, p_2).$$

It is routine to check that (A, E', ω', U') is a continuous action of $\mathcal{H} \ltimes P$.

Let $(A^i, E^i, \omega^i, U^i)$ for $i = 1, 2$ be two continuous actions of $\mathcal{G} \ltimes P$ and let (F, V) be a transformation between them. Construct actions $((A^i)', (E^i)', (\omega^i)', (U^i)')$ of $\mathcal{H} \ltimes P$ for $i = 1, 2$ as above. The induced transformation (F', V') between these actions of $\mathcal{H} \ltimes P$ is given by $F' = F$ and

$$\begin{aligned} V'_{(t,\eta,p)} = V_{t,p}\colon (E^1)'_{(t,\eta,p)} \otimes_{A^1_p} F_p = E^1_{(t,p)} \otimes_{A^1_p} F_p \\ \xrightarrow{V_{t,p}} F_{\alpha_t(p)} \otimes_{A^2_{\alpha_t(p)}} E^2_{(t,p)} = F_{\alpha_t(p)} \otimes_{A^2_{\alpha_t(p)}} (E^2)'_{(t,\eta,p)}. \end{aligned}$$

This is indeed a transformation, and it remains an equivalence if (F, V) is one. Thus equivalent actions of $\mathcal{G} \ltimes P$ induce equivalent actions of $\mathcal{H} \ltimes P$, as asserted by Theorem 7.4.5.

There is nothing to do to transfer modifications between $\mathcal{G} \ltimes P$ and $\mathcal{H} \ltimes P$.

Strict actions of crossed modules of topological groupoids are defined in [28]. We make this explicit in the case we need:

Definition 7.4.6. Let A be a $C_0(P)$-C^*-algebra. A strict action of $\mathcal{H} \ltimes P$ on A by automorphisms is given by continuous group homomorphisms $\alpha\colon H^1 \to \operatorname{Aut}(A)$ and $u\colon H^2 \to U(M(A))$ with $\alpha_{\partial(h)} = \operatorname{Ad}_{u(h)}$ for all $h \in H^2$ and $\alpha_h(u_k) = u_{c_h(k)}$ for all $h \in H^1$, $k \in H^2$, such that the homomorphism $C_0(P) \to ZM(A)$ is H^1-equivariant, where H^1 acts on P through the quotient map $H^1 \to \mathbb{R}^n$, $(t, \eta) \mapsto t$.

Such a strict action of $\mathcal{H} \ltimes P$ by automorphisms induces an action by correspondences. First, we take $E = r^*(A)$ as a left Hilbert A-module, with the right action of $s^*(A)$ through α: $(x \cdot a)(t, \eta, p) = x(t, \eta, p) \cdot \alpha_{t,\eta}(a(t, \eta, p))$ for all $x \in E$, $a \in s^*(A)$; so $a(t, \eta, p) \in A_p$ and $\alpha_{t,\eta}(a(t, \eta, p)) \in A_{\alpha_t(p)} = A_{r(t,p)} \ni x(t, \eta, p)$, as it should be. The isomorphisms ω map $x_1 \otimes x_2 \mapsto x_1 \cdot \alpha_{t,\eta}(x_2)$ for $x_1 \in E_{(t,\eta,p)}$, $x_2 \in E_{(t_2,\eta_2,p_2)}$, and $U_{((\theta,\xi),(t,\eta),p)}\colon E_{(t,\eta,p)} \to E_{(t,\eta+\theta,p)}$ multiplies on the right with the unitary $u(\theta, \xi)^*$. This is an action by correspondences as defined above.

Theorem 7.4.7. *Any action of $\mathcal{H} \ltimes P$ by correspondences is equivalent to an action that comes from a strict action of $\mathcal{H} \ltimes P$ by automorphisms.*

Proof. This is contained in [29, Thm. 5.3]. □

By construction, the action by correspondences coming from a strict action has $E \cong r^*(A)$ as a left Hilbert A-module and $E \cong s^*(A)$ as a right Hilbert A-module. Therefore, if F is a C^*-correspondence between A^1 and A^2 for two such actions $(A^1, E^1, \omega^1, U^1)$, then $E^1 \otimes_{s^*(A^1)} s^*(F) \cong s^*(F)$ as a right Hilbert $s^*(A)$-module and $r^*(F) \otimes_{r^*(A^2)} E^2 \cong r^*(F)$ as a left Hilbert $r^*(A)$-module. In particular, these are isomorphisms of Banach spaces. The isomorphism V in a transformation is therefore given by bounded linear maps $V_g\colon F_{s(g)} \to F_{r(g)}$; the extra conditions to induce an isomorphism of correspondences $E^1 \otimes_{s^*(A^1)} s^*(F) \to r^*(F) \otimes_{r^*(A^2)} E^2$ are exactly the usual equivariance conditions for an equivariant correspondence. Thus the notion of equivalence for actions of $\mathcal{H} \ltimes P$ by correspondences amounts to a standard notion of equivariant Morita equivalence.

Theorem 7.4.8. *Let A be a C^*-algebra. A strict action of $\mathcal{H}$ on A is equivalent to a pair of continuous maps $\bar\alpha\colon \mathbb{R}^n \to \operatorname{Aut}(A)$ and $\bar u\colon \Lambda^2\mathbb{R}^n \to UM(A)$ with the following properties:*

(a) $\bar\alpha_s \bar\alpha_t \bar\alpha_{-s-t} = \operatorname{Ad}_{\bar u(s\wedge t)}$ for all $s, t \in \mathbb{R}^n$ and $\bar\alpha_0 = \mathrm{id}$;

(b) $\bar u$ is a group homomorphism;

(c) $\bar\alpha_s(\bar u(t \wedge v))\bar u(t \wedge v)^* \in UM(A)$ is central and fixed by $\bar\alpha_w$ for all elements $s, t, v, w \in \mathbb{R}^n$;

(d) $\bar{\alpha}_s(\bar{u}(t\wedge v))\,\bar{u}(t\wedge v)^* = \bar{u}(s\wedge v)\,\bar{\alpha}_t(\bar{u}(s\wedge v))^*$ *for all* $s,t,v\in\mathbb{R}^n$.

Proof. Let $\bar{\alpha}$ and $\bar{u}$ have the required properties. We claim that there are well-defined group homomorphisms

$$\begin{aligned} \alpha\colon H^1 &\to \operatorname{Aut}(A), & (t,\eta) &\mapsto \operatorname{Ad}_{\bar{u}(\eta)}\circ\bar{\alpha}_t,\\ u\colon H^2 &\to UM(A), & (\theta, s\wedge\theta_2) &\mapsto \bar{u}(\theta)\cdot\bar{\alpha}_s(\bar{u}(\theta_2))\bar{u}(\theta_2)^*. \end{aligned}$$

The map α is clearly well-defined. Since $\bar{\alpha}_s(\bar{u}(\eta))\bar{u}(\eta)^*$ is central, the automorphisms $\operatorname{Ad}_{\bar{u}(\eta)}$ and $\operatorname{Ad}_{\bar{\alpha}_s(\bar{u}(\eta))}$ are equal for all $s\in\mathbb{R}^n$, $\eta\in\Lambda^2\mathbb{R}^n$. Hence

$$\begin{aligned} &\alpha(t_1,\eta_1)\circ\alpha(t_2,\eta_2) = \operatorname{Ad}_{\bar{u}(\eta_1)}\circ\bar{\alpha}_{t_1}\circ\operatorname{Ad}_{\bar{u}(\eta_2)}\circ\bar{\alpha}_{t_2}\\ &= \operatorname{Ad}_{\bar{u}(\eta_1)}\circ\operatorname{Ad}_{\bar{\alpha}_{t_1}(\bar{u}(\eta_2))}\circ\bar{\alpha}_{t_1}\circ\bar{\alpha}_{t_2}\\ &= \operatorname{Ad}_{\bar{u}(\eta_1)}\circ\operatorname{Ad}_{\bar{u}(\eta_2)}\circ\operatorname{Ad}_{\bar{u}(t_1\wedge t_2)}\circ\bar{\alpha}_{t_1+t_2} = \operatorname{Ad}_{\bar{u}(\eta_1)\bar{u}(\eta_2)\bar{u}(t_1\wedge t_2)}\circ\bar{\alpha}_{t_1+t_2}\\ &= \operatorname{Ad}_{\bar{u}(\eta_1+\eta_2+t_1\wedge t_2)}\circ\bar{\alpha}_{t_1+t_2} = \alpha(t_1+t_2,\eta_1+\eta_2+t_1\wedge t_2). \end{aligned}$$

Thus α is a homomorphism. We have $u(\theta,\xi) = u(\theta,0)\,u(0,\xi)$ and $\theta\mapsto u(\theta,0)$ is a group homomorphism. By assumption, the map $(s,t,v)\mapsto u(0,s\wedge t\wedge v)$ is antisymmetric and additive in t and v; thus it is additive also in s and hence descends to a group homomorphism on $\Lambda^3\mathbb{R}^n$. Since $u(0,s\wedge t\wedge v)$ is central, u is a well-defined group homomorphism. We have $\alpha_{(t,\eta)}(u_{\theta,\xi}) = u_{\theta,\xi+t\wedge\theta}$ and $\alpha_{(0,\theta)} = \operatorname{Ad}_{u(\theta,\xi)}$ by construction, so α and u form a strict action of the crossed module $\mathcal{H}$. Conversely, such a strict action (α,u) gives back $(\bar{\alpha},\bar{u})$ as above by taking $\bar{\alpha}_t = \alpha_{t,0}$ and $\bar{u}_\eta = u_{\eta,0}$. □

Remark 7.4.9. Crossed products for crossed module actions are studied in [28] in the strict case, and in [30] for actions by correspondences. This construction is, however, not very useful in our case because the crossed product is simply zero whenever the associator U is non-trivial. More precisely, the analysis in [30] shows that the crossed product factors through the quotient of A by the ideal generated by $(U_\xi - 1)a$ for all $a\in A$, $\xi\in\Lambda^3\mathbb{R}^n$. In the interesting case where the associator is needed, this quotient is zero.

To get a non-zero C^*-algebra, we may first tensor A by some other action B of $\mathcal{H}$ with the opposite associator, so that the diagonal action of $\mathcal{H}$ on $A\otimes B$ has trivial associator. Thus $A\otimes B$ carries a Green twisted action of $\mathbb{R}^n$, and the crossed product with $\mathcal{H}$ on $A\otimes B$ is the appropriate $\mathbb{R}^n$-crossed product for such twisted actions. We do not know, however, how to choose B so that $(A\otimes B)\rtimes\mathcal{H}$ could play the role of the usual crossed product by $\mathbb{R}^n$ in T-duality.

7.5 Computing the lifting obstruction

Let P be a $\mathbb{T}^n$-space with orbit space X and let A be a continuous trace C^*-algebra over P. We have seen that the $\mathbb{T}^n$-action on P lifts to an $\mathbb{R}^n$-action on A if and only if a certain tricharacter $\chi\colon \Lambda^3\mathbb{R}^n \to C(X,\mathbb{R})$ vanishes. How can we compute χ?

Let $p \in P$, let $1 \leq i < j < k \leq n$, and let $\mathbb{T}_{ijk} \subseteq \mathbb{T}^n$ be the three-dimensional subtorus given by the coordinates i, j, k. The value of χ at $e_i \wedge e_j \wedge e_k$ and the orbit of p may also be computed by the smaller system where we replace $\mathbb{T}^n$, P and A by $\mathbb{T}_{ijk}$, $\mathbb{T}_{ijk} \cdot p \subseteq P$, and the restriction of A to the orbit $\mathbb{T}_{ijk} \cdot p$; this is because the lifting obstruction χ is natural. Thus it suffices to compute the lifting obstruction in the case of a transitive action of a three-dimensional torus $\mathbb{T}^3$ on a space P, with some continuous trace C^*-algebra A over P. If the stabilisers in P are not discrete, then P is two-dimensional, so $H^3(P,\mathbb{Z}) = 0$ and A is a trivial bundle of compact operators. In this case, the $\mathbb{T}^n$-action clearly lifts to a $\mathbb{T}^n$-action on A, so the lifting obstruction vanishes. So we may assume that the stabilisers of points in P are discrete.

We replace the action of $\mathbb{T}^3$ by one of $\mathbb{R}^3$, which we want to lift to A. This action is still transitive, and by assumption the stabiliser of a point is a lattice $\Gamma \subseteq \mathbb{R}^3$. We choose coordinates in $\mathbb{R}^n$ so that this lattice becomes $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. Thus we are reduced to computing the lifting obstruction in the case where $P = \mathbb{R}^3/\mathbb{Z}^3$ with the standard $\mathbb{R}^3$-action.

Our theory tells us that the continuous trace C^*-algebra carries an action of the crossed module of Lie groupoids $\mathcal{H} \ltimes P$. Since $\mathbb{R}^3$ acts transitively on P, this crossed module is equivalent to the $\mathcal{H}$-stabiliser of a point in P; for general $n \in \mathbb{N}$ this is the crossed module of groups $\tilde{\mathcal{H}}$ with $\tilde{H}^1 = \mathbb{Z}^n \times \Lambda^2\mathbb{R}^n$ and $\tilde{H}^2 = \Lambda^2\mathbb{R}^n \oplus \Lambda^3\mathbb{R}^n$ with the restrictions of ∂ and c from $\mathcal{H}$. In our situation, the usual construction of induced actions of groups generalises to crossed modules. As we will see, any action of $\mathcal{H} \ltimes P$ on our continuous trace C^*-algebra A is induced from an action of $\tilde{\mathcal{H}}$, namely, the restriction of the action of $\mathcal{H} \ltimes P$ on a single fibre $A_p = \mathbb{K}$.

A strict action of $\tilde{\mathcal{H}}$ on $\mathbb{K}$ consists of two group homomorphisms $\tilde{\alpha}\colon \tilde{H}^1 \to \mathrm{Aut}(\mathbb{K})$ and $\tilde{u}\colon \tilde{H}^2 \to U(M(\mathbb{K}))$ such that $\tilde{\alpha}_{\partial(h)} = \mathrm{Ad}_{\tilde{u}(h)}$ and $\tilde{u}(c_g(h)) = \tilde{\alpha}_g(\tilde{u}(h))$ for $g \in \tilde{H}^1$, $h \in \tilde{H}^2$. Let

$$A = \mathrm{Ind}_{\tilde{H}^1}^{H^1}(\mathbb{K}) = \{f \in C_b(H^1,\mathbb{K}) \mid f(\tilde{g}\cdot g) = \tilde{\alpha}_{\tilde{g}}(f(g)) \text{ for all } \tilde{g} \in \tilde{H}^1,\ g \in H^1\}$$

be the induced C^*-algebra. By [113, Cor. 6.21] it has continuous trace with spectrum $H^1/\tilde{H}^1 = \mathbb{T}^n$. It carries a canonical action $\alpha\colon H^1 \to \mathrm{Aut}(A)$ given by $\alpha_h(f)(g) = f(g\cdot h)$. Let $u\colon H^2 \to U(M(A))$ be defined by $u_k(g) = \tilde{u}(c_g(k))$ for $g \in H^1$ and $k \in H^2$. The pair (α, u) defines a strict action of $\mathcal{H} \ltimes P$ on A in the sense of Definition 7.4.6. The proof is straightforward, using that $\partial(H^2) \subset \tilde{H}^1$ lies in the centre of H^1.

Similarly, an equivalence (F, V) between two strict actions $(\tilde{\alpha}, \tilde{u})$ and $(\tilde{\alpha}', \tilde{u}')$ induces an equivalence between the induced actions. Since any action of $\tilde{\mathcal{H}}$ on A by correspondences is equivalent to a strict one by Theorem 7.4.7, induction descends to a well-defined group homomorphism $\mathrm{Ind}\colon \mathrm{Br}(\tilde{\mathcal{H}}) \to \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)$. Restricting an action to the fibre yields another group homomorphism $\mathrm{Res}\colon \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \to \mathrm{Br}(\tilde{\mathcal{H}})$.

Lemma 7.5.1. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be the crossed modules described above acting on a stable continuous trace algebra A with spectrum $\mathbb{T}^n$. Restriction to the fibre of A and induction of a strict action to $\mathcal{H} \ltimes \mathbb{T}^n$ are inverse to each other and yield a group isomorphism $\mathrm{Br}(\tilde{\mathcal{H}}) \cong \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)$.*

Proof. If we restrict the induced action to the fibre over 0, we regain the action on $\mathbb{K}$ we started with. Thus $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}_{\mathrm{Br}(\tilde{\mathcal{H}})}$.

Let $p_0 = 0 \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let (α, u) be a strict action of $\mathcal{H} \ltimes \mathbb{T}^n$ on A. Let $(\tilde{\alpha}, \tilde{u})$ be the restricted action of $\tilde{H}^1$ on $A(p_0)$, which we identify with $\mathbb{K}$. Let $A' = \mathrm{Ind}_{\tilde{H}^1}^{H^1}(\mathbb{K})$ and denote the induced action by (α', u'). Consider the $C_0(P)$-algebra homomorphism $\varphi\colon A \to A'$, which maps $a \in A$ to f_a with $f_a(g) = \alpha_g(a)(p_0)$. Since $f_{\alpha_h(a)}(g) = f_a(gh)$, it is equivariant. It is norm-preserving and therefore injective. Local triviality and a partition of unity argument show that it is also surjective, hence an isomorphism. The extension of φ to the multiplier algebra maps u to u'. Therefore, φ is an isomorphism that intertwines the two actions of $\mathcal{H} \ltimes \mathbb{T}^n$, which implies $\mathrm{Ind} \circ \mathrm{Res} = \mathrm{id}_{\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)}$. □

To compute $\mathrm{Br}(\tilde{\mathcal{H}})$ we may further simplify the situation by weakening. The equivalence between $\mathcal{H}$ and $\mathcal{G}$ restricts to one between $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{G}}$, where $\tilde{G}^1 = \mathbb{Z}^n$, $\tilde{G}^2 = \Lambda^3\mathbb{R}^n$ with the restriction of the bigroupoid structure from $\mathcal{G}$. Since $\mathbb{K}$ is equivalent to $\mathbb{C}$, strict actions of $\tilde{\mathcal{H}}$ on $\mathbb{K}$ by automorphisms are equivalent to actions of $\tilde{\mathcal{G}}$ by correspondences on $\mathbb{C}$.

Such an action has $A = \mathbb{C}$ and $E_g = \mathbb{C}$ for all $g \in \tilde{G}^1 = \mathbb{Z}^n$ because this is, up to isomorphism, the only imprimitivity bimodule from $\mathbb{C}$ to itself. The multiplication maps and the action of bigons give maps $\omega\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ and $U\colon \Lambda^3\mathbb{R}^n \to \mathbb{T}$ because an isomorphism $\mathbb{C} \to \mathbb{C}$ is simply multiplication by a scalar of modulus one. The conditions for an action require the following:

(i) $\omega(0, l) = \omega(k, 0) = 1$ for all $k, l \in \mathbb{Z}^n$;

(ii) U is a continuous group homomorphism;

(iii) $\omega(k + l, m)\omega(k, l)U(k \wedge l \wedge m) = \omega(k, l + m)\omega(l, m)$ for all $k, l, m \in \mathbb{Z}^n$.

For a discrete group Γ and a Γ-module M, we denote the group cohomology of Γ with coefficients in M by $H^k_{\mathrm{gr}}(\Gamma, M)$.

Theorem 7.5.2. *Let $k = \binom{n}{3}$. The pair (ω, U) associated to an action of $\tilde{\mathcal{G}}$ on $\mathbb{C}$ has the following properties:*

(a) U is trivial on $\Lambda^3\mathbb{Z}^n$ and therefore yields a character on $\mathbb{T}^k = \Lambda^3\mathbb{R}^n/\Lambda^3\mathbb{Z}^n$,

(b) ω is a group 2-cocycle representing a central extension of $\mathbb{Z}^n$ by $\mathbb{T}$.

Moreover, we have group isomorphisms

$$\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong \mathrm{Br}(\tilde{\mathcal{H}}) \cong \mathrm{Br}(\tilde{\mathcal{G}}) \cong \mathbb{Z}^k \times H^2_{gr}(\mathbb{Z}^n, \mathbb{T}).$$

Proof. It follows from (c) in the conditions listed above that the restriction of U to $\Lambda^3\mathbb{Z}^n$ is the coboundary of ω. In particular, it represents the trivial element in cohomology. This is impossible if U is a non-trivial tricharacter on $\mathbb{Z}^n$. Thus ω is a 2-cocycle classifying a central $\mathbb{T}$-extension of $\mathbb{Z}^n$ and U is a character on $\Lambda^3\mathbb{R}^n/\Lambda^3\mathbb{Z}^n \cong \mathbb{T}^k$. The group of all such characters is the Pontrjagin dual of $\mathbb{T}^k$, which is $\mathbb{Z}^k$.

An equivalence (F, V) between two actions (ω, U) and (ω', U') has to have $F = \mathbb{C}$, and $V\colon \mathbb{Z}^n \to \mathbb{T}$ has to satisfy $V(0) = 1$ and $\omega(k,l)V(k+l) = V(l)V(k)\,\omega'(k,l)$ for all $k, l \in \mathbb{Z}^n$. Therefore (ω, U) and (ω', U') are equivalent if and only if $U = U'$ and ω differs from ω' by a coboundary. Moreover, the product of elements in the Brauer group of the bigroupoid $\tilde{\mathcal{G}}$ translates into the product of characters and cocycles. Since conditions (a)–(c) above fully characterise an action of $\tilde{\mathcal{G}}$ on $\mathbb{C}$ by correspondences, every pair (ω, U) is associated to such an action. Altogether, we have constructed a group isomorphism $\mathrm{Br}(\tilde{\mathcal{G}}) \to \mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$. The isomorphism $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong \mathrm{Br}(\tilde{\mathcal{H}})$ was constructed in Lemma 7.5.1, and $\mathrm{Br}(\tilde{\mathcal{G}}) \cong \mathrm{Br}(\tilde{\mathcal{H}})$ follows from the equivalence of the two bigroupoids. □

We can make the inverse of the isomorphism $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ more explicit: We first have to transfer the action of $\tilde{\mathcal{G}}$ to $\tilde{\mathcal{H}}$ using the functor π, then make that weak action strict by passing to a stabilisation by [29, Thm. 5.3]. We have $U_{(\theta,\xi)} = U^{\tilde{\mathcal{G}}}_{\xi}$ for $\theta \in \Lambda^2\mathbb{R}^n$, $\xi \in \Lambda^3\mathbb{R}^n$ because $\pi(\theta, \xi) = \xi$. For $k_1, k_2 \in \mathbb{Z}^n$, $\eta_1, \eta_2 \in \Lambda^2\mathbb{R}^n$, the natural transformation $\pi(k_1, \eta_1)\cdot\pi(k_2, \eta_2) \Rightarrow \pi((k_1, \eta_1)\cdot(k_2, \eta_2))$ in $\tilde{\mathcal{G}}$ is $k_1 \wedge \eta_2$, so $\omega((k_1, \eta_1), (k_2, \eta_2)) = U^{\tilde{\mathcal{G}}}_{k_1\wedge\eta_2} \cdot \omega^{\tilde{\mathcal{G}}}(k_1, k_2)$.

To turn this action of the crossed module $\tilde{\mathcal{H}}$ by correspondences into a strict action by automorphisms, we take the Hilbert space of L^2-sections of the Fell bundle $(\mathbb{C})_{g\in\tilde{H}^1}$ over $\tilde{H}^1$. In our case, this is simply $\mathcal{K} = L^2(\mathbb{Z}^n \times \Lambda^2\mathbb{R}^n)$. The multiplication in the Fell bundle given by ω forms an action of $\tilde{\mathcal{H}}$ on this Hilbert module over $\mathbb{C}$.

We give $\mathbb{K}(\mathcal{K})$ the induced action, so that $\mathcal{K}$ is an equivariant Morita equivalance between $\mathbb{K}(\mathcal{K})$ and $\mathbb{C}$ with the given action of $\tilde{\mathcal{H}}$. Since the action of $\tilde{\mathcal{H}}$ on $\mathbb{C}$ is non-trivial, $\mathcal{K}$ is not a Hilbert space representation of $\tilde{\mathcal{H}}$, but only a projective Hilbert space representation.

On $\tilde{H}^1$, this is given by

$$((k_2, \eta_2) \cdot f)(k_1, \eta_1) = \omega((k_1, \eta_1), (k_2, \eta_2)) f(k_1 + k_2, \eta_1 + \eta_2 + k_1 \wedge k_2)$$

for $k_1, k_2 \in \mathbb{Z}^n$ and $\eta_1, \eta_2 \in \Lambda^2 \mathbb{R}^n$. This induces an action $\alpha\colon \tilde{H}^1 \to \mathrm{Aut}(\mathbb{K}(\mathcal{K}))$. Together with the homomorphism $u\colon \tilde{H}^2 \to U(\mathcal{K})$ given by

$$u_{(\theta,\xi)}(f)(k, \eta) = U^{\tilde{\mathcal{G}}}(\xi + k \wedge \theta)\, f(k, \eta + \theta)$$

this is the strict action of $\tilde{\mathcal{H}}$ on $\mathbb{K}(\mathcal{K})$ that corresponds to the action by correspondences of $\tilde{\mathcal{G}}$ on $\mathbb{C}$ given by $(\omega^{\tilde{\mathcal{G}}}, U^{\tilde{\mathcal{G}}})$.

Finally, we induce the action of $\tilde{\mathcal{H}}$ on $\mathbb{K}(\mathcal{K})$ to $\mathcal{H} \ltimes \mathbb{T}^n$. As described above, this produces an action of $\mathcal{H} \ltimes P$ with $P = \mathbb{R}^n/\mathbb{Z}^n$. More precisely, we get an action of $\mathcal{H} \ltimes P$ on a continuous trace C^*-algebra over P.

Theorem 7.5.3. *Let A be the continuous trace C^*-algebra that corresponds to the element $(1, 0) \in \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^3, \mathbb{T}) \cong \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^3)$. Then the Dixmier–Douady invariant $\mathrm{dd}(A) \in H^3(\mathbb{T}^3, \mathbb{Z})$ is a generator.*

Proof. Let $(0, x) \in \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^3, \mathbb{T})$ and choose a cocycle $\omega\colon \mathbb{Z}^3 \times \mathbb{Z}^3 \to \mathbb{T}$ representing x. The action of $\tilde{\mathcal{H}}$ on $\mathbb{C}$ by correspondences associated to this pair is pulled back from an action of $\mathbb{Z}^3$ on $\mathbb{C}$ by correspondences via the canonical functor $\tilde{\mathcal{H}} \to \mathbb{Z}^3$. Let $\mathcal{K}' = L^2(\mathbb{Z}^3)$. The group $\mathbb{Z}^3$ acts projectively on $\mathcal{K}'$ via ω. This induces an honest representation $\alpha\colon H^1 \to \mathbb{Z}^3 \to \mathrm{Aut}(\mathbb{K}(\mathcal{K}'))$ of H^1 on the compact operators. Let $u\colon H^2 \to U(M(\mathbb{K}(\mathcal{K}')))$ be the trivial homomorphism. The pair (α, u) is a strict action of $\tilde{\mathcal{H}}$ on $\mathbb{K}(\mathcal{K}')$. By the same reasoning as above, we may choose A to be the C^*-algebra obtained by inducing this $\tilde{\mathcal{H}}$-action to an $\mathcal{H}$-action. Since $A \cong C(\mathbb{T}^3, \mathbb{K}(\mathcal{K}'))$, its Dixmier–Douady class vanishes. But $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^3) \to \mathrm{Br}(\mathbb{T}^3) \cong H^3(\mathbb{T}^3, \mathbb{Z})$ is surjective, therefore $(1, 0)$ has to be mapped to a generator of $H^3(\mathbb{T}^3, \mathbb{Z}) \cong \mathbb{Z}$. □

Corollary 7.5.4. *Let $k = \binom{n}{3}$. The group homomorphism*

$$\mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}) \cong \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \to H^3(\mathbb{T}^n, \mathbb{Z}),$$

which maps an element $([U], [\omega])$ to the Dixmier–Douady class of the associated continuous trace C^*-algebra restricts to an isomorphism $\mathbb{Z}^k \to H^3(\mathbb{T}^n, \mathbb{Z})$ and maps $H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ to zero.

Proof. Each projection $p_{ijk}\colon \mathbb{T}^n \to \mathbb{T}^3$ for $1 \leq i < j < k \leq n$ induces a commutative diagram

$$\begin{tikzcd} \mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}) \arrow[r] & H^3(\mathbb{T}^n, \mathbb{Z}) \\ \mathbb{Z} \times H^2_{\mathrm{gr}}(\mathbb{Z}^3, \mathbb{T}) \arrow[r] \arrow[u, "p^*_{ijk}"] & H^3(\mathbb{T}^3, \mathbb{Z}). \arrow[u, "p^*_{ijk}"'] \end{tikzcd}$$

Since the vertical arrows are split by corresponding inclusions $\mathbb{T}^3 \to \mathbb{T}^n$, they are injective. In particular, $p^*_{ijk}(1) \in \{\pm e_i \wedge e_j \wedge e_k\} \subset \Lambda^3\mathbb{Z}^n \cong \mathbb{Z}^k$. The statement now follows from Theorem 7.5.3 because $\mathbb{Z}^k \times H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ is generated by the images of all p^*_{ijk}. $\square$

7.6 Crossed module actions and T-duality

Let A be a continuous trace C^*-algebra whose spectrum is a $\mathbb{T}^n$-space P with orbit space X. A second such pair A' with spectrum P' and the same orbit space is said to be (topologically) *T-dual* to A if the $\mathbb{R}^n$-action on P lifts to one on A with $A' \cong A \rtimes \mathbb{R}^n$. In particular, this implies an isomorphism of twisted K-groups $K_*(A) \cong K_{*-n}(A')$ by the Connes–Thom isomorphism. In the case of principal $\mathbb{T}^1$-bundles, any pair (A, P) has a unique T-dual.

A T-dual need no longer exist for higher-dimensional torus bundles. The first obstruction against it is the lifting obstruction discussed above. Even if it vanishes, the crossed product need not be a continuous trace C^*-algebra. Whether this is the case is determined by a class in $H^1(X, \mathbb{Z}^\ell)$ for $\ell = \binom{n}{2}$, which is derived from the Mackey obstruction as in [97, Thm. 3.1]. If it does not vanish, the crossed product turns out to be a bundle of non-commutative tori.

A (non-associative) Fell bundle (E, ω, U) given by an action of $\mathcal{H} \ltimes P$ on A by correspondences combines all of the obstructions into one structure: We have already identified the lifting obstruction. By Theorem 7.5.2, the Brauer group of the fibre $\mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n) \cong H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}) \times H^3(\mathbb{T}^n, \mathbb{Z})$ may be interpreted as the group of all possible Dixmier–Douady classes and Mackey obstructions of $\mathbb{T}^n$. As described in [109], the group $H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T})$ can be equipped with a natural topology, and [109, Lem. 3.3] gives a homomorphism

$$M\colon \mathrm{Br}_{\mathcal{H}}(P) \to C(X, H^2_{gr}(\mathbb{Z}^n, \mathbb{T}))\ ,$$

which sends $[A] \in \mathrm{Br}_{\mathcal{H}}(P)$ to the function that maps x to the Mackey obstruction of $[A(x)] \in \mathrm{Br}_{\mathcal{H}}(\mathbb{T}^n)$. The homotopy class $[M(A)] \in \pi_0(C(X, H^2_{\mathrm{gr}}(\mathbb{Z}^n, \mathbb{T}))) \cong H^1(X, \mathbb{Z}^\ell)$ for $\ell = \binom{n}{2}$ vanishes if and only if there is a classical T-dual, see [97, Lem. 3.1].

To summarise: The lifting obstruction of the Fell bundle (E, ω, U) vanishes if and only if the multiplication in the Fell bundle is associative. If it is, then we can form the "section" C^*-algebra associated to the Fell bundle. This will again be of continuous trace if and only if $[M(A)]$ vanishes. If it does, then the section algebra is the classical T-dual of A, else it is a non-commutative T-dual. As can be seen from this, the non-associative Fell bundle obtained from the crossed module action contains all information of the T-dual in case it exists and all residual information in case it does not. Therefore it should be considered the more fundamental object.

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