Equivariant higher Dixmier-Douady theory for circle actions

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A lightning-quick reminder about C^*-algebras

Let H be a Hilbert space (in most examples, H will be separable). Denote by $\mathcal{B}(H)$ the bounded operators on H.

Definition

An algebra A is called a C^***-algebra** if it is a norm-closed $*$-subalgebra of $\mathcal{B}(H)$.

Example

- $C_0(X)$ for a locally compact Hausdorff space X,
- $\mathbb{K} = \mathbb{K}(H)$, ie. the compact operators on H,
- $C_0(X, A)$, ie. functions on X with values in the C^*-algebra A,
- $M_n^{\otimes\infty}$, ie. the infinite tensor product of $M_n(\mathbb{C})$.

Operator K-theory

- A - unital C^*-algebra
- $\mathcal{P}_{\mathbb{C}}(A)$ - isomorphism classes of finitely generated projective right Hilbert A-modules

Note that $\mathcal{P}_{\mathbb{C}}(A)$ is a monoid with respect to $\oplus$.

Definition

The 0th K-group of a unital C^*-algebra A is defined as

$$K_0(A) = \mathrm{Gr}(\mathcal{P}_{\mathbb{C}}(A), \oplus)$$

Properties:

- $A \mapsto K_n(A)$ exists for all $n \in \mathbb{Z}$ and also for non-unital C^*-algebras.
- Bott periodicity: $K_i(A) \cong K_{i+2}(A)$

Operator K-theory - Some examples

First observations:

- $K_n(C_0(X)) \cong K^n(X)$ (Serre-Swan theorem),
- $K_0(\mathbb{C}) \cong \mathbb{Z}$,
- $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$, in general $K_0(A) \cong K_0(A \otimes \mathbb{K})$.

What is $K_0(D)$ for $D = M_n(\mathbb{C})^{\otimes\infty}$?

$$M_n(\mathbb{C})^{\otimes k} \xrightarrow{\cdot \otimes 1_n} M_n(\mathbb{C})^{\otimes(k+1)} \quad \rightsquigarrow \quad \text{on } K_0: \quad \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$$

Since K_0 is continuous, we have

$$K_0(D) \cong \varinjlim (\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) \cong \mathbb{Z}[\tfrac{1}{n}]$$

Classification of C^*-algebras

Theorem (Kirchberg-Phillips)

Infinite-dimensional separable, nuclear, simple, unital and purely infinite C^*-algebras in the UCT class are classified up to isomorphism by their K-theory and the class of the unit in K_0.

Definition

The **infinite Cuntz algebra $\mathcal{O}_\infty$** is the universal C^*-algebra generated by countably many partial isometries s_i satisfying the relation

$$s_i^* s_j = \delta_{i,j} 1 \,.$$

Theorem

Let A be a simple, separable, nuclear C^*-algebra. Then $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite.

Representation of $\mathcal{O}_\infty$

Let H be a separable ∞-dim. Hilbert space. Pick an orthonormal basis $\{e_i\}_{i\in\mathbb{N}} \subset H$. Let

$$\mathcal{F}(H) = \bigoplus_{k=0}^{\infty} H^{\otimes k}$$

(where $H^{\otimes 0} = \mathbb{C}$). The closed $*$-subalgebra of $\mathcal{B}(\mathcal{F}(H))$ generated by

$$s_i(\xi) = e_i \otimes \xi \quad \text{and} \quad s_i^*(v \otimes \xi) = \langle e_i, v\rangle \xi$$

is isomorphic to $\mathcal{O}_\infty$.

Classification of C^*-algebras (contd.)

Theorem (many hands, 2015)

Simple, separable, unital, infinite dimensional, nuclear C^*-algebras with finite nuclear dimension in the UCT class are classified by their ordered K-theory, class of unit in K_0 and trace data.

The nuclear dimension is a generalisation of the covering dimension.

Theorem (Castillejos, Evington, Tikuisis, White, Winter)

A simple, separable, unital, infinite dimensional, nuclear C^*-algebra A has finite nuclear dimension if and only if $A \cong A \otimes \mathcal{Z}$.

The algebra $\mathcal{Z}$ is called the Jiang-Su algebra. It is a direct limit of dimension-drop algebras.

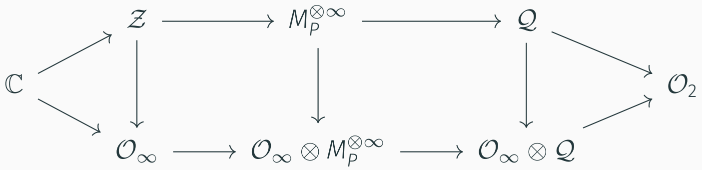
Strongly self-absorbing C^*-algebras

Definition (Toms-Winter)

A separable, unital C^*-algebra D is called **strongly self-absorbing** if $\exists$ an isomorphism $\psi\colon D \to D \otimes D$ and a continuous path $u\colon [0,1) \to U(D \otimes D)$ with

$$\lim_{t\to 1} \|\psi(d) - u_t(d \otimes 1_D)u_t^*\| = 0 \ .$$

Strongly self-absorbing C^*-algebras in the UCT-class:



Definition

A (locally trivial) **fibre bundle** $E \to X$ with **fibre** F consists of

- a topological space E, called the **total space**,
- a continuous map $\pi \colon E \to X$

with the following property:

- each point $x \in X$ has an open neighbourhood $U \subset X$ such that there exists a homeomorphism $\varphi_U \colon U \times F \to E|_U$ that makes

$$\begin{array}{ccc} U \times F & \xrightarrow{\varphi_U} & E|_U \\ & {\scriptstyle \mathrm{pr}_U} \searrow \quad \swarrow {\scriptstyle \pi|_U} & \\ & U & \end{array}$$

commute. The map φ_U is called a **trivialisation** over U.

Homotopy classification of fibre bundles

Theorem

Let G be a topological group. There exists a topological space BG (called ***classifying space*** *of G) and a principal G-bundle*

$$EG \to BG$$

that has the following property:

For every compact Hausdorff space X and every principal G-bundle $E \to X$ there exists a continuous map $f : X \to BG$ (unique up to homotopy) such that

$$\begin{array}{ccc} E & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

is a pullback diagram.

In other words...

Corollary

Let $\mathcal{B}un_G(X)$ be the set of isomorphism classes of fibre bundles with structure group G. There is a natural $1:1$-correspondence

$$\mathcal{B}un_G(X) \leftrightarrow [X, BG]$$

Examples of classifying spaces:

- $B\mathbb{Z} \simeq S^1$, hence $[X, B\mathbb{Z}] \cong [X, S^1] \cong H^1(X, \mathbb{Z})$.
- Hermitian line bundles have structure group $U(1)$ and

$$[X, BU(1)] \cong H^2(X, \mathbb{Z}) \ .$$

Bundles with fibre $\mathbb{K}$ - Classical Dixmier-Douady theory

Definition

Let B be a C^*-algebra. A **bundle of C^*-algebras** $\mathcal{A} \to X$ with fibre B is a fibre bundle with structure group $\mathrm{Aut}(B)$ (equipped with the point-norm topology). Let $\mathcal{B}un_B(X)$ be the isomorphism classes.

Let $\mathbb{K}$ be compact operators on an ∞-dim. sep. Hilbert space. All $*$-automorphisms of $\mathbb{K}$ are generalised inner, ie.

$$\mathrm{Aut}(\mathbb{K}) \cong PU(H) := U(H)/U(1) \ .$$

By long exact sequence of π_k and since $U(H)$ is contractible

$$\pi_k(B\mathrm{Aut}(\mathbb{K})) \cong \begin{cases} \mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases}$$

Corollary (Dixmier-Douady)

There is a natural group isomorphism:

$$\delta\colon \mathcal{B}un_{\mathbb{K}}(X) \to [X, BPU(H)] \cong H^3(X, \mathbb{Z})$$

called the ***Dixmier-Douady class****. This isomorphism is multiplicative in the sense that*

$$\delta(\mathcal{K}_1 \otimes \mathcal{K}_2) = \delta(\mathcal{K}_1) + \delta(\mathcal{K}_2) \ .$$

Let $\mathcal{K}^{op}$ be the bundle of compacts with reversed multiplication, then

$$\delta(\mathcal{K}^{op}) = -\delta(\mathcal{K}) \ .$$

Why does this work and how can we generalise it?

Note that we have an isomorphism

$$\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$$

$\mathcal{B}un_{\mathbb{K}}(X)$ inherits a *monoid structure* via fibrewise tensor product.

Key idea

Try to get similar results with

- a strongly self-absorbing C^*-algebra D instead of $\mathbb{K}$,
- or with $D \otimes \mathbb{K}$.

Topological properties of strongly self-absorbing C^*-algebras:

- $\mathrm{Aut}(D)$ is contractible (and so is $B\mathrm{Aut}(D)$),
- $K_0(D)$ is a ring (and $K_1(D) = 0$ if D satisfies the UCT),

Unit spaces of ring spectra

Observation: $PU(H) \simeq BU(1)$

object	classifying space
hermitian line bundle L	$PU(H) \simeq BU(1)$
bundle of compact operators $\mathcal{A}$	$BPU(H) \simeq BBU(1)$

$U(1)$ is an abelian group $\quad \Rightarrow \quad BU(1), BBU(1), \dots$ exist

Observation: Line bundles form a subgroup in $GL_1(K^0(X))$.
Idea: Extend the above table to all of $GL_1(K^0(X))$!

object	classifying space
virtual line bundles	$GL_1(KU)$
?	$BGL_1(KU)$

$GL_1(KU)$ is an infinite loop space $\quad \Rightarrow \quad BGL_1(KU)$ exists

Higher Dixmier-Douady Theory (jt. with M. Dadarlat)

Theorem (Dadarlat-P.)

Let D be a strongly self-absorbing C^*-algebra and let KU^D be the ring spectrum representing K-theory with coefficients $K_i(D)$. Then:

$$\mathcal{B}un_{D\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU^D)] ,$$
$$\mathcal{B}un_{\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU)] ,$$
$$\mathcal{B}un_{M_{n^\infty}\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU[\tfrac{1}{n}])_+] .$$

Remarks:

- We also determined the homotopy type of $B\mathrm{Aut}(D \otimes \mathbb{K})$ for all strongly self-absorbing C^*-algebras D,
- The group $[X, BGL_1(KU)]$ and its variants can be determined via the Atiyah-Hirzebruch spectral sequence.

A higher twist over $SU(n)$ (jt. with D. Evans)

Input:

- $G = SU(n)$,
- exponential functor $F \colon (\mathcal{V}ect_{\mathbb{C}}^{iso}, \oplus) \to (\mathcal{V}ect_{\mathbb{C}}^{iso}, \otimes)$.

Output:

- infinite tensor product $D = \mathrm{End}(F(\mathbb{C}^n))^{\otimes \infty}$,
- C^*-algebra $C^*(\mathcal{E})$ (arising from a Fell bundle $\mathcal{E} \to \mathcal{G}$).

Construction generalises the basic gerbe that plays a crucial role in work of Freed, Hopkins and Teleman.

Theorem (Evans-P.)

$$C^*(\mathcal{E}) \otimes \mathbb{K} \cong C(G, \mathcal{A}) \ ,$$

where $\mathcal{A} \to G$ is a bundle with fibre $D \otimes \mathbb{K}$.

Equivariant operator K-theory

- G - compact Lie group,
- A - unital C^*-algebra with continuous G-action $\alpha \colon G \to \mathrm{Aut}(A)$,
- $\mathcal{P}^G_{\mathbb{C}}(A)$ - analogous to $\mathcal{P}_{\mathbb{C}}(A)$, but Hilbert A-modules come equipped with a G-action $\lambda \colon G \to \mathcal{B}(E)$ with

$$\lambda_g(\xi \cdot a) = \lambda_g(\xi) \cdot \alpha_g(a) \quad \text{and} \quad \langle \lambda_g(\xi), \lambda_g(\eta) \rangle_A = \alpha_g(\langle \xi, \eta \rangle_A)$$

Definition

The 0th K-group of a unital C^*-algebra A is defined as

$$K_0^G(A) = \mathrm{Gr}(\mathcal{P}^G_{\mathbb{C}}(A), \oplus)$$

Equivariant operator K-theory (contd.)

Properties:

- $A \mapsto K_n^G(A)$ exists for all $n \in \mathbb{Z}$ and also for non-unital G-C^*-algebras.
- Bott periodicity: $K_i^G(A) \cong K_{i+2}^G(A)$
- $K_0^G(\mathbb{C}) \cong \mathrm{Rep}(G)$

What is $K_0^G(D)$ for $D = \mathrm{End}(V)^{\otimes\infty}$?

$$\mathrm{End}(V)^{\otimes k} \xrightarrow{\cdot\otimes 1} \mathrm{End}(V)^{\otimes(k+1)} \quad \rightsquigarrow \quad \text{on } K_0^G:\ \mathrm{Rep}(G) \xrightarrow{\cdot V} \mathrm{Rep}(G)$$

Hence, $K_0^G(D) \cong \varinjlim\left(\mathrm{Rep}(G) \xrightarrow{\cdot V} \mathrm{Rep}(G)\right) \cong \mathrm{Rep}(G)[V^{-1}]$

When life hands you Lie groups...

...restrict to the maximal torus.

- $G = SU(n)$
- have G-equivariant bundle $\mathcal{A} \to G$ with fibre $D \otimes \mathbb{K}$

The pullback $\mathcal{A}|_{\mathbb{T}^{n-1}} \to \mathbb{T}^{n-1}$ to the maximal torus $\mathbb{T}^{n-1} \subset SU(n)$ has trivial action on base space.

Simplified setting

- $G = \mathbb{T}^1 = \mathbb{T}$ (circle group),
- $D = \mathrm{End}(V)^{\otimes\infty}$ for $\mathbb{T}$-representation V,
- $\mathbb{K} = \mathbb{K}(H)$ for $H = \ell^2(\mathbb{Z}) \otimes H_0$, where $u_z(\delta_k) = z^k\,\delta_k$, $\dim(H_0) = \infty$ and H_0 separable.

Question: What is the homotopy type of $B\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$?

Equivariant higher Dixmier-Douady theory

Theorem (Evans-P.)

1. $\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$ *is an infinite loop space with respect to* $\otimes$,
2. *The associated cohomology theory* $E^*_{D,\mathbb{T}}(X)$ *satisfies*

$$E^0_{D,\mathbb{T}}(X) = [X, \mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})] \quad \textit{and} \quad E^1_{D,\mathbb{T}}(X) \cong [X, B\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})] \ ,$$

3. $E^k_{\mathbb{C},\mathbb{T}}(X) \cong H^k(X,\mathbb{Z}) \oplus H^{k+2}(X,\mathbb{Z})$ *and in particular*

$$E^1_{\mathbb{C},\mathbb{T}}(X) \cong H^3_{\mathbb{T}}(X,\mathbb{Z}) \cong Br_{\mathbb{T}}(X) \ ,$$

4. $E^k_{D,\mathbb{T}}(pt) \cong \pi_{-k}(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K}))$ *(see next slides).*

The homotopy groups of $\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$

- $\mathbb{T}$-representation $V \quad \rightsquigarrow \quad p_V(t) \in \mathbb{Z}[t, t^{-1}] \cong \mathrm{Rep}(\mathbb{T})$,
- may assume without loss of generality that $p_V(t) \in \mathbb{Z}[t]$,
- $K_0^{\mathbb{T}}(D) \cong \mathbb{Z}[t, t^{-1}, p_V(t)^{-1}]$.

Definition

The bounded subring of $\mathbb{Z}[t, t^{-1}, p_V(t)^{-1}]$ is defined as

$$\mathrm{R}_{\mathrm{bdd}} = \left\{ q \in \mathbb{Z}[t, p_V(t)^{-1}] \mid -m \leq q \leq m \text{ for some } m \in \mathbb{N} \right\}$$

Moreover, let

$$\mathrm{R}^0_{\mathrm{bdd}} = \{ r \in \mathrm{R}_{\mathrm{bdd}} \mid r(0) = 0 \} ,$$

$$\mathrm{R}^\infty_{\mathrm{bdd}} = \left\{ \frac{q}{p_V^k} \in \mathrm{R}_{\mathrm{bdd}} \mid q \in \mathbb{Z}[t], k \geq 0, \deg(q) < kd \right\}$$

The homotopy groups of $\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$ (contd.)

Suppose that $p_V(t) = \sum_{i=0}^{d} a_i t^i$. Then

Theorem

We have $\pi_{2k-1}(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})) = 0$ *and*

$$\pi_0(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})) \cong GL_1(K_0^{\mathbb{T}}(D))_+ \ ,$$

$$\pi_2(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})) \cong \mathrm{R}_{bdd} \ ,$$

$$\pi_{2k}(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})) \cong \begin{cases} \mathrm{R}_{bdd} & \text{if } (a_0 > 1) \text{ and } (a_d > 1) \ , \\ \mathrm{R}^0_{bdd} & \text{if } (a_0 = 1) \text{ and } (a_d > 1) \ , \\ \mathrm{R}^\infty_{bdd} & \text{if } (a_0 > 1) \text{ and } (a_d = 1) \ , \\ \mathrm{R}^0_{bdd} \cap \mathrm{R}^\infty_{bdd} & \text{if } (a_0 = 1) \text{ and } (a_d = 1) \ , \end{cases}$$

Reason: $\pi_{2k}(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})) \cong \pi_{2k-1}(U(D^{\mathbb{T}}))$.

Understanding $E^1_{D,\mathbb{T}}(X)$

Theorem (Evans-P.)

$$E^1_{D,\mathbb{T}}(\mathbb{T}^n) \cong H^1(\mathbb{T}^n, G) \oplus H^3(\mathbb{T}^n, \mathsf{R}_{bdd}) \oplus \bigoplus_{k=2}^{\infty} H^{2k+1}(\mathbb{T}^n, \mathsf{R}^{0,\infty}_{bdd}) \ .$$

with $G = GL_1(K_0^{\mathbb{T}}(D))_+$.

The map $\theta \colon \mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K}) \to K_0^{\mathbb{T}}(D)$ given by

$$\theta(\alpha) = [\alpha(1 \otimes e)] \in K_0((D \otimes \mathbb{K})^{\mathbb{T}}) \cong K_0^{\mathbb{T}}(D)$$

factors through a homomorphism to $GL_1(K_0^{\mathbb{T}}(D))_+$ and is an isomorphism on π_0. It induces a natural transformation

$$\theta_* \colon E^1_{D,\mathbb{T}}(X) \to H^1(X, GL_1(K_0^{\mathbb{T}}(D))_+) \ .$$

Understanding $E^1_{D,\mathbb{T}}(X)$ for $D = \mathbb{C}$

Reminder: $E^1_{\mathbb{C},\mathbb{T}}(X) \cong \mathrm{Br}_{\mathbb{T}}(X) \cong H^1(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z})$

- For $D = \mathbb{C}$ we have $K_0^{\mathbb{T}}(\mathbb{C}) \cong \mathbf{Rep}(\mathbb{T}) \cong \mathbb{Z}[t, t^{-1}]$.
- Therefore $GL_1(K_0^{\mathbb{T}}(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ and $GL_1(K_0^{\mathbb{T}}(\mathbb{C}))_+ \cong \mathbb{Z}$.

Hence,

$$\theta_* \colon E^1_{\mathbb{C},\mathbb{T}}(X) \to H^1(X,\mathbb{Z}) \ .$$

This is the Phillips-Raeburn obstruction:

$$\mathbb{K}\text{-bundle } \mathcal{A} \to X \ \rightsquigarrow \ \mathbb{K}^{\mathbb{T}}\text{-bundle } \mathcal{A}^{\mathbb{T}} \to X \ \rightsquigarrow \ \mathbb{Z}\text{-bundle } \widehat{\mathcal{A}^{\mathbb{T}}} \to X \ .$$

where $\mathbb{K}^{\mathbb{T}} \cong C_0(\mathbb{Z}) \otimes \mathbb{K}(H_0)$ and $\widehat{\mathcal{A}^{\mathbb{T}}} \to X$ takes the spectrum fibrewise.

$$F \colon E^1_{\mathbb{C},\mathbb{T}}(X) \to H^3(X,\mathbb{Z})$$

gives the Dixmier-Douady class after forgetting the $\mathbb{T}$-action.

What is next?

- In the non-equivariant case we have an equivalence of infinite loop spaces $\mathrm{Aut}(M_n(\mathbb{C})^{\otimes\infty} \otimes \mathbb{K}) \cong GL_1(KU[1/n])_+$.

  What about $\mathbf{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$?
- How is all of this related to equivariant stable homotopy theory?
- Is $\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$ coming from a fixed-point spectrum?
- What about non-trivial group actions on X?
- There are other interesting group actions on strongly self-absorbing C^*-algebras.

  Can we say anything about $\mathrm{Aut}_G(D \otimes \mathbb{K})$ then?

Thank you!