Unit spectra of K-theory via strongly self-absorbing C^*-algebras

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Thm.: Let A be a strongly self-abs. C^*-algebra, then

$$B\,\mathrm{Aut}(A \otimes \mathcal{O}_\infty \otimes \mathbb{K}) \simeq BGL_1(KU^A).$$

Def.: commutative symmetric ring spectrum

- sequence R_n, $n \in \mathbb{N}_0$ of ptd. top. spaces w/ Σ_n-action (basept.-pres.)
- pointed eq. maps: $\eta_n : S^n = \underbrace{S^1 \wedge \ldots \wedge S^1}_{n \text{ times}} \longrightarrow R_n$
- pointed $\Sigma_m \times \Sigma_n$-equiv. maps

$$\mu_{m,n} : R_m \wedge R_n \longrightarrow R_{m+n}$$

s.th. the following hold

(a) associativity of $\mu_{m,n}$

(b) $\mu_{p,q} \circ (\eta_p \wedge \eta_q) = \eta_{p+q}$

(c) commutativity of $\mu_{m,n}$

$$\begin{array}{ccc} R_m \wedge R_n & \xrightarrow{\mu_{m,n}} & R_{m+n} \\ \downarrow{\scriptstyle tw} & & \downarrow{\scriptstyle \tau_{m,n}} \\ R_n \wedge R_m & \xrightarrow{\mu_{n,m}} & R_{n+m} \end{array}$$

block permutation

Idea: Generalize

$$\begin{array}{ccc} \text{Rings} & \longrightarrow & \text{AbGrps} \\ R & \longmapsto & GL_1(R) \end{array}$$ to Spectra.

Def: $\mathcal{I}$-spaces (a la Sagave & Schlichtkrull)

$\mathcal{I}$ is the cat. with obj.: $\underline{n} = \{1, \ldots, n\}$ and mor.: injective maps ($\underline{0} = \emptyset$ is initial)

$\mathcal{I}$ is symmetric monoidal ($\underline{m} \sqcup \underline{n}$ via concatenation)

An $\mathcal{I}$-space is a functor $X : \mathcal{I} \longrightarrow \mathrm{Top}_*$.

An $\mathcal{I}$-monoid is an $\mathcal{I}$-space + natural transformation

$$\mu : X \times X \longrightarrow X \circ \sqcup \quad \text{of } \mathcal{I}^2\text{-spaces.}$$

+ associativity, unitality.

<u>Construction</u> (Schlichtkrull): commutative $\mathcal{I}$-monoid $X \rightsquigarrow$ Γ-space $\Gamma(X)$ with underlying space $X_{h\mathcal{I}}$ $\rightsquigarrow$ connective spectrum if X is grouplike

underlying space of an $\mathcal{I}$-space X is $X_{h\mathcal{I}} = \underset{\mathcal{I}}{\mathrm{hocolim}}\, X$

If X is <u>convergent</u>, this agrees with the telescope construction

($\exists$ unbdd., non-dec. seq. of nat. numbers λ_n, $n \geq 0$, s.th. $X(n_1) \longrightarrow X(n_2)$ is λ_n-connected for $n_1, n_2 \geq n$.)

R commutative symmetric ring spectrum

$\mathcal{I}$-space: $(\Omega^\infty R)^*(\underline{n}) := \{ f : S^n \longrightarrow R_n \mid \exists\, g : S^m \longrightarrow R_m$ s.th.
$\mu_{n,m}\circ(f \wedge g)$ and $\mu_{m,n}\circ(g\wedge f)$
are both homotopic to $\eta_{n+m} \}$

Exercise: Check that this is an $\mathcal{I}$-space.

The multiplication

$$(\Omega^\infty R)^*(\underline{n}) \times (\Omega^\infty R)^*(\underline{m}) \longrightarrow (\Omega^\infty R)^*(\underline{n \sqcup m})$$
$$(f_1, f_2) \longmapsto \mu_{n,m}\circ(f_1 \wedge f_2)$$

turns this into a commutative $\mathcal{I}$-monoid.

space of units: $GL_1(R) = \operatorname{hocolim}_{\mathcal{I}} (\Omega^\infty R)^*$

If R convergent, then $\pi_0(GL_1(R)) = GL_1(\pi_0 R)$.

$gl_1(R)$ spectrum associated to $\Gamma((\Omega^\infty R)^*)$.

Def.: A separable, unital C^*-algebra A is called strongly self-absorbing, if $\exists$ iso $\psi : A \longrightarrow A\otimes A$ and a path $u : [0,1) \longrightarrow U(A\otimes A)$ s.th.

$$\lim_{t\to 1} \| u_t\, \psi(a)\, u_t^* - \ell(a) \| = 0 \quad \forall a \in A,$$

where $\ell(a) = a \otimes 1$.

Properties: .) $K_*(A)$ is a ring with unit $[1_A] \in K_0(A)$

.) $X \longmapsto K_*(C_0(X)\otimes A)$ is a multiplicative cohomology theory on locally cpt. spaces.

.) $\mathrm{Aut}(A)$ is contractible. (Dadarlat, P.)

Associate an $\mathcal{I}$-space to A via:

$$G_A(\underline{n}) = \mathrm{Aut}((A\otimes \mathbb{K})^{\otimes n})$$

$\mathcal{I}$-monoid:

$$G(\underline{n}) \times G(\underline{m}) \longrightarrow G(\underline{n} \sqcup \underline{m})$$
$$(\alpha, \beta) \longmapsto \alpha \otimes \beta$$

Def.: An EH-$\mathcal{I}$-group is an $\mathcal{I}$-monoid in TopGrp s.th.

$$\begin{tikzcd}
G(\underline{n}) \times G(\underline{n}) \times G(\underline{m}) \times G(\underline{m}) \arrow[r, "(\mu_{n,m}\times\mu_{n,m})\circ\tau"] \arrow[d, "\nu_n \times \nu_m"'] \arrow[dr, phantom, "\#"] & G(\underline{n}\sqcup\underline{m}) \times G(\underline{n}\sqcup\underline{m}) \arrow[d, "\nu_{n\sqcup m}"] \\
G(\underline{n}) \times G(\underline{m}) \arrow[r, "\mu_{n,m}"'] & G(\underline{n}\sqcup\underline{m})
\end{tikzcd}$$

commutes.

Two Γ-spaces : $\underline{n} \rightsquigarrow B_\nu G(\underline{n})$ is a commutative $\mathcal{I}$-monoid

$\rightsquigarrow \Gamma(B_\nu G)$

and $B_\mu \Gamma(G)$.

<u>Thm</u> (Dadarlat, P.) G convergent EH-$\mathcal{I}$-group s.th. $G(\underline{n})$ has htype of a CW-cplx

Then there is a strict equiv. of very special Γ-spaces

$$B_\mu \Gamma(G) \simeq \Gamma(B_\nu G). \quad \text{(given by a zig-zag of h-eq.)}$$

In our case : Whenever we have $\underline{n} \longrightarrow \underline{m}$ with $n \neq 0$, $G_A(\underline{n}) \longrightarrow G_A(\underline{m})$ is a h-eq. since A is strongly self-absorbing!

$$\Rightarrow \quad B\,Aut(A \otimes \mathbb{K}) = BG_A(1) \xrightarrow{\simeq} (B_\nu G_A)_{h\mathcal{I}}.$$

<u>A spectrum representing $X \mapsto K_*(C_0(X) \otimes A)$</u>

graded C^*-algebras : $\hat{S} = C_0(\mathbb{R})$ with grading by even/odd fcts.

$\mathbb{C}l_1$ Clifford algebra gen. by $1, e$ with $e^2 = 1$

$= \mathbb{C}[\mathbb{Z}/2]$.

<u>Facts</u> : ·) $\hat{S}$ is a coalgebra

$$\Delta : \hat{S} \longrightarrow \hat{S} \otimes \hat{S}, \quad \varepsilon : \hat{S} \longrightarrow \mathbb{C} \quad + \text{coass.} \ldots$$

·) $\underbrace{\mathbb{C}l_1 \hat{\otimes} \ldots \hat{\otimes} \mathbb{C}l_1}_{n\text{-times}} \cong \mathbb{C}l_n$

<u>Def</u> : $KU^A_n := hom_{gr}(\hat{S}, (\mathbb{C}l_1 \hat{\otimes} (A \otimes \mathbb{K}))^{\hat{\otimes} n})$.

·) ptd. by 0-hom.

·) Σ_n-action on KU^A_n (permuting the factors with signs!)

·)
$$KU^A_n \wedge KU^A_m \longrightarrow KU^A_{n+m}$$
$$(\varphi \wedge \psi) \longmapsto (\varphi \hat{\otimes} \psi) \circ \Delta$$

·) $\eta_1 : S^1 \longrightarrow KU^A_1$ corresponds to an element in

$$\hat{\eta}_1 \in hom_{gr}(\hat{S}, \underbrace{C_0(S^1, pt)}_{\cong C_0(\mathbb{R}) \text{ ungraded}} \hat{\otimes} \mathbb{C}l_1 \hat{\otimes} (A \otimes \mathbb{K}))$$

gives rise to $\hat{\eta}_n \in hom_{gr}(\hat{S}, C_0(\mathbb{R}^n) \hat{\otimes} (\mathbb{C}l_1 \hat{\otimes} (A \otimes \mathbb{K}))^{\hat{\otimes} n})$

<u>Thm</u> (Dadarlat, P.) : For a strongly self-absorbing C^*-algebra, the spaces KU^A_n form a comm. symm. ring spectrum representing $X \mapsto K_*(C(X) \otimes A)$, which is convergent.

We have a map of $\mathcal{I}$-spaces

$$\begin{array}{ccc} G_A(\underline{n}) & \longrightarrow & \Omega^\infty (KU^A)^*(\underline{n}) \\ \| & & \| \\ \mathrm{Aut}((A \otimes \mathbb{K})^{\otimes n}) & & \hom_{gr}\left(\hat{S},\, C_0(\mathbb{R}^n) \otimes (\mathbb{C}l_1 \hat{\otimes} (A \otimes \mathbb{K}))^{\hat{\otimes} n}\right) \\ & & \ni \end{array}$$

by letting α act on $\hat{\eta}_n$.

This turns out to be a map of commutative $\mathcal{I}$-monoids, i.e. it induces

$$\Gamma(G_A \cancel{(\underline{n})}) \longrightarrow \Gamma(\Omega^\infty (KU^A)^*) \qquad (*)$$

in particular

$$\Gamma(B_\nu G_A) \xrightarrow{\cong} B_\mu \Gamma(G_A) \longrightarrow B_\mu \Gamma(\Omega^\infty (KU^A)^*)$$

$$B\,\mathrm{Aut}(A \otimes \mathbb{K}) \longrightarrow BGL_1(KU^A)$$

$(*)$ is an iso on all homotopy groups except on π_0 where it is induced by the inclusion

$$K_0(A)_+^\times \hookrightarrow K_0(A)^\times .$$