<u>Unit spectra of K-theory via strongly self-absorbing C^*-algebras</u>

jt. work w/ Marius Dadarlat

units of top. K-theory : $GL_1(KU) \simeq BU_\otimes \times \mathbb{Z}/2$

$[X, GL_1(KU)]$: virtual vector bdls of virtual dim. ± 1 for X cpt. Hausdorff

$GL_1(KU)$ is an infinite loop space

<u>Question</u>: What does $BGL_1(KU)$ classify?

$$\mathrm{Pic}(KU) \supset BGL_1(KU) \supset BSL_1(KU) = BBU_\otimes \overset{\text{splits off}}{\supset} BBU(1) = K(\mathbb{Z},3)$$

The $K(\mathbb{Z},3)$-part has many nice interpretations:

- gerbes
- projective Hilbert bundles
- bundles of cpt. operators, in fact we have a map

$$Aut(\mathbb{K}) \longrightarrow GL_1(KU) \quad \text{delooping to} \quad BAut(\mathbb{K}) \to BGL_1(KU)$$

$\rightsquigarrow$ Bundles of C^*-algebras?

<u>C^*-algebras</u>

- $B(H)$, $\mathbb{K} = \mathbb{K}(H)$, $M_n\mathbb{C}$
- inductive limits of matrix algebras, e.g.

$$A_1 = \mathbb{C}, \quad A_n = A_{n-1} \otimes M_n\mathbb{C}, \quad \varphi_n : A_{n-1} \longrightarrow A_n, \quad a \longmapsto a \otimes 1$$

$$\rightsquigarrow \varphi_{m,n} : A_m \longrightarrow A_n, \quad n \geq m$$

$\varinjlim A_n \subset \prod A_n / \bigoplus A_n$ generated by elements $(\varphi_{m,n}(a_m))_{n \in \mathbb{N}_{\geq m}}$ for all $a_m \in A_m$

Morally, in this case infinite tensor product

$$\mathbb{C} \otimes M_2\mathbb{C} \otimes M_3\mathbb{C} \otimes M_4\mathbb{C} \otimes \dots = M_{\mathbb{Q}}$$

Likewise

$$\cancel{\mathbb{C}\otimes} M_2\mathbb{C} \otimes M_2\mathbb{C} \otimes M_2\mathbb{C} \otimes \dots = M_{2^\infty} \cong \text{CAR-algebra}$$

- the infinite Cuntz-algebra $\mathcal{O}_\infty$

  H sep. ∞-dim. Hilbert space

  $$\mathcal{T}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}, \quad \xi \in H \rightsquigarrow S_\xi : \mathcal{T}(H) \longrightarrow \mathcal{T}(H), \quad S_\xi(\eta) = \xi \otimes \eta$$

  C^*-algebra gen. by S_ξ, S_ξ^* for $\|\xi\| = 1$ in $B(\mathcal{T}(H))$ is $\mathcal{O}_\infty$

<u>Def</u>.: A separable unital C^*-algebra A is called <u>strongly self-absorbing</u> (Toms, Winter) if $\exists\, \psi : A \xrightarrow{\cong} A \otimes A$ and $u : [0,1) \longrightarrow U(A \otimes A)$ s.th.

$$\lim_{t \to 1} \| u_t\, \psi(a)\, u_t^* - \ell(a) \| = 0 \quad \text{for } \ell(a) = a \otimes 1.$$

<u>Ex.</u>: $\mathbb{C}$, $M_{\mathbb{Q}}$, $\mathcal{O}_\infty$, M_{p^∞}, $\mathcal{O}_2$, $\mathcal{Z}$, tensor products ...

<u>Properties</u>:
- $K_0(A)$ is a ring with unit $[1_A]$
- $X \longmapsto K_*(C_0(X) \otimes A)$ is a multiplicative cohomology theory on loc. cpt. Hausdorff spaces
- There is a tensor product on $\mathrm{Aut}(A \otimes \mathbb{K})$ (and on $\mathrm{Aut}(A)$):

  Choose iso $\psi : A \otimes \mathbb{K} \longrightarrow (A \otimes \mathbb{K})^{\otimes 2}$

  $$\mu_\psi : \mathrm{Aut}(A \otimes \mathbb{K}) \times \mathrm{Aut}(A \otimes \mathbb{K}) \longrightarrow \mathrm{Aut}(A \otimes \mathbb{K})$$
  $$(\alpha, \beta) \longmapsto \psi^{-1} \circ (\alpha \otimes \beta) \circ \psi$$

  This is a <u>group hom.</u>!

  $$\rightsquigarrow B\mu_\psi : B\,\mathrm{Aut}(A \otimes \mathbb{K}) \times B\,\mathrm{Aut}(A \otimes \mathbb{K}) \longrightarrow B\mathrm{Aut}(A \otimes \mathbb{K})$$

<u>Homotopy Groups of $\mathrm{Aut}(A \otimes \mathbb{K})$</u>

$\mathrm{Aut}_0(A \otimes \mathbb{K})$ component of $\mathrm{id}_{A \otimes \mathbb{K}}$

$\mathrm{Proj}_0(A \otimes \mathbb{K})$ component of $1 \otimes e$ for a rank 1-projection e

$\mathrm{Aut}_{1\otimes e}(A \otimes \mathbb{K})$ stabilizer of $1 \otimes e$

$$\mathrm{Aut}_{1\otimes e}(A \otimes \mathbb{K}) \longrightarrow \mathrm{Aut}_0(A \otimes \mathbb{K}) \longrightarrow \mathrm{Proj}_0(A \otimes \mathbb{K}) \quad \text{is a principal bdl.}$$
$$\alpha \longmapsto \alpha(1 \otimes e)$$

<u>Thm.</u> (Dadarlat, P.): A strongly self-abs. Then $Aut_{1\otimes e}(A \otimes \mathbb{K})$ is contractible (point-norm top.)

<u>Proof</u>: Construct a continuous map

$$\Psi : [0,1] \longrightarrow Hom(A\otimes \mathbb{K}, (A\otimes\mathbb{K})^{\otimes 2})$$

s.th. $\psi(0) = \ell$, $\psi(1) = r$, $\psi(t)(1\otimes e) = 1\otimes e \otimes 1 \otimes e$

and $\psi(t)$ iso. for $t \in (0,1)$.

$$H(t,\alpha) = \begin{cases} \alpha & \text{for } t=0 \\ \psi(t)^{-1}\circ(\alpha\otimes id)\circ\psi(t) & \text{for } 0<t<1 \\ id & \text{for } t=1 \end{cases}$$ □

<u>Cor.</u>: $Aut_0(A\otimes\mathbb{K}) \simeq Proj_0(A\otimes\mathbb{K}) \underset{\substack{\uparrow \\ \text{Thm.}}}{\simeq} BU(A)$

for $k>0$:

$\Rightarrow \pi_k(Aut(A\otimes\mathbb{K})) = \pi_k(Aut_0(A\otimes\mathbb{K})) = \pi_{k-1}(U(A)) \underset{\substack{\uparrow \\ \text{Thm. by Jiang.}}}{=} K_k(A)$.

<u>Thm</u> (Dadarlat, P.) A str. self-abs., Then $Aut(A)$ is contractible.

For π_0: $Aut(A\otimes\mathbb{K}) \longrightarrow Proj(A\otimes\mathbb{K})$ maps onto "invertible" projections

$\Longrightarrow \pi_0(Aut(A\otimes\mathbb{K})) = K_0(A)_+^{\times}$.

<u>Thm</u> (Dadarlat, P.): A str. self-abs. C^*-alg. Then:

·) $B\,Aut(A\otimes\mathbb{K})$ is an infinite loop space
H-space structure μ

·) $B\,Aut(\mathcal{O}_\infty\otimes A\otimes\mathbb{K}) \simeq BGL_1(KU^A)$
as infinite loop spaces.

KU^A is a (symmetric) spectrum representing $X \longmapsto K_*(C_0(X)\otimes A)$.

·) $KU^{\mathbb{C}} \simeq KU^{\mathcal{O}_\infty} \simeq KU^{\mathcal{Z}} \simeq KU$

·) $KU^{M_{p^\infty}} \simeq KU[\frac{1}{p}]$

·) $KU^{M_{\mathbb{Q}}}$ rationalization

<u>Idea of the proof</u>: μ_ψ is a group hom. $\rightsquigarrow$ Eckmann-Hilton like arg.

Two deloopings: $B_\nu \,Aut(A\otimes \mathbb{K})$ as a group

$B_\mu \,Aut(A\otimes \mathbb{K})$ w.r.t. μ_ψ

$$\begin{aligned} B_\nu \,Aut(A\otimes \mathbb{K}) &\simeq \Omega B_\mu B_\nu \,Aut(A\otimes \mathbb{K}) \\ &\overset{\text{"}}{\simeq} \Omega B_\nu B_\mu \,Aut(A\otimes \mathbb{K}) \\ &\simeq B_\mu \,Aut(A\otimes \mathbb{K}) \end{aligned}$$

KU^A carries an action of $Aut(A\otimes \mathbb{K})$

$\rightsquigarrow Aut(A\otimes \mathbb{K}) \longrightarrow GL_1(KU^A)$ by acting on the (multiplicative) unit.

$$B_\nu \,Aut(A\otimes \mathbb{K}) \xrightarrow{\simeq} B_\mu \,Aut(A\otimes \mathbb{K}) \longrightarrow B_\mu GL_1(KU^A)$$

Check iso on π_k. ◼

<u>Applications</u>

•) Rational characteristic classes of continuous fields / bundles

$$B\,Aut(A\otimes \mathbb{K}) \longrightarrow B\,Aut(\mathcal{O}_\infty \otimes M_{\mathbb{Q}} \otimes A \otimes \mathbb{K}) \simeq \prod_{k=1}^{\infty} K(\mathbb{Q}, 2k+1) \times K(\mathbb{Q}^\times, 1)$$

$\delta_k(\mathcal{A}) \in H^{2k+1}(X, \mathbb{Q})$, $\delta_0(\mathcal{A}) \in H^1(X, \mathbb{Q})$

:) Torsion elements in $[X, BBU_\otimes]$ or $[X, BSL_1(KU^A)]$:

$B\,Aut(M_n A) \longrightarrow B\,Aut_0(A\otimes \mathbb{K})$ all torsion elements come from bundles of matrix algebras

•) "Higher" twisted K-theory

$K_*(C_0(X, \mathcal{A}))$ for $\mathcal{A} \to X$ bdl. w/ fiber $A\otimes \mathcal{O}_\infty \otimes \mathbb{K}$

•) 2-vector bdls.